

On Index Matrices

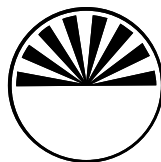
Krassimir Atanassov



**Prof. Marin Drinov Publishing House
of Bulgarian Academy of Sciences**

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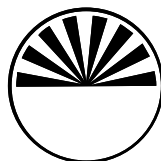
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BULGARIAN ACADEMY OF SCIENCES

Krassimir T. Atanassov

On Index Matrices

Sofia • 2025



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Foreword

The concept of index matrices originated more than forty years ago. My first attempt to formalize it was inspired by a conversation with the distinguished Bulgarian physicist, Academician Hristo Hristov. In May 1987, he invited me to present my ideas at his seminar at the Institute of Nuclear Research and Nuclear Energy of the Bulgarian Academy of Sciences. After the seminar, he encouraged me to prepare a manuscript, and to his surprise, I brought him a completed version the very next morning. By the end of that year, my first paper on index matrices was published.

Over the following 25 years, a number of applications of index matrices were discovered and developed. They proved to be a convenient tool for describing transitions in generalized nets—extensions of Petri nets, which I have studied for over 45 years—as well as for representing graphs, relationships between fuzzy and intuitionistic fuzzy sets, decision-making procedures, cognitive maps, intercriteria analysis, and related areas.

In 2013, Prof. Janusz Kacprzyk Academician of the Polish, Bulgarian and a lot of other Academies of Sciences and editor of several Springer book series, encouraged me to collect these results into a monograph, which was published by Springer in 2014 under the title “Index Matrices: Towards an Augmented Matrix Calculus” [20]. Since then, further theoretical developments have emerged, along with new applications of index matrices. In particular, the concept of three-dimensional and, more generally, n -dimensional index matrices was introduced, together with the corresponding basic operations, although challenges remain—especially with respect to the multiplication operation.

This book presents selected results in this area. For brevity, most statements are given without proofs, which can be found in the cited literature.

To illustrate the motivation behind index matrices, consider the following example (cf. [20]). Suppose that we are given two matrices:

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} 10 & 11 \\ 12 & 13 \\ 14 & 15 \end{vmatrix}.$$

A natural question is whether their sum can be defined. Since the matrices are of sizes 2×3 and 3×2 , respectively, this operation is not defined in classical matrix calculus. Nevertheless, such a question may still arise in practice, suggesting the possibility of extending the original problem to a closely related, well-posed one. For instance, by augmenting the matrices to a common size, several plausible extensions can be constructed. In fact, there are nine such possibilities, each corresponding to a minimal modification of the original problem:

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} + \begin{vmatrix} 10 & 11 \\ 12 & 13 \\ 14 & 15 \end{vmatrix} \approx \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 0 \end{vmatrix} + \begin{vmatrix} 10 & 11 & 0 \\ 12 & 13 & 0 \\ 14 & 15 & 0 \end{vmatrix} = \begin{vmatrix} 11 & 13 & 3 \\ 16 & 18 & 6 \\ 14 & 15 & 0 \end{vmatrix}$$

or

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} + \begin{vmatrix} 10 & 11 \\ 12 & 13 \\ 14 & 15 \end{vmatrix} \approx \begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 4 & 5 & 6 \end{vmatrix} + \begin{vmatrix} 10 & 11 & 0 \\ 12 & 13 & 0 \\ 14 & 15 & 0 \end{vmatrix} = \begin{vmatrix} 11 & 13 & 3 \\ 12 & 13 & 0 \\ 18 & 20 & 0 \end{vmatrix}$$

or

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} + \begin{vmatrix} 10 & 11 \\ 12 & 13 \\ 14 & 15 \end{vmatrix} \approx \begin{vmatrix} 0 & 0 & 0 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} + \begin{vmatrix} 10 & 11 & 0 \\ 12 & 13 & 0 \\ 14 & 15 & 0 \end{vmatrix} = \begin{vmatrix} 10 & 11 & 0 \\ 13 & 15 & 3 \\ 18 & 20 & 6 \end{vmatrix}$$

or ...

These constructions are not strictly equivalent to the original ill-posed problem, which is why the symbol “ $\approx$ ” is used to indicate approximate or analogous formulations. In this way, an ill-posed matrix operation is replaced by a closely related, well-defined one.

To formalize the above reasoning and to devise suitable means for handling such problems, over the past 40 years I have developed the concept of the index matrix and its corresponding extended matrix calculus, enabling these ideas to be applied in a mathematically consistent and rigorous way.

This book presents the main results on index matrices, along with a number of open problems.

I express my sincere gratitude to Peter Vassilev, Veselina Bureva, and Vassia Atanassova for their insightful discussions that helped shape this book. I also gratefully acknowledge the support of the Bulgarian National Science Fund under Grant No. KP-06-N-72/8 (14 December 2023).

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Krassimir T. Atanassov

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Index Matrices (IM): Definitions and examples

In this chapter, following and extending the results from [20], we give definition of the concept of an Index Matrix (IM) and of the operations over it. More precise, this type of IM should be called 2-Dimensional IMs (2DIMs). Examples in the area of number theory are use as illustrations.

1.1 Definition of an IM

Let $\mathcal{I}$ be a fixed set of indices. In some cases it is suitable to use more complex indices. So, let us define for the natural number $k \geq 1$:

$$\mathcal{I}^k = \{\langle i_1, i_2, \dots, i_k \rangle \mid (\forall j : 1 \leq j \leq k)(i_j \in \mathcal{I})\}$$

and

$$\mathcal{I}^* = \bigcup_{1 \leq k \leq \infty} \mathcal{I}^k.$$

Let $\mathcal{X}$ be a fixed set of some objects. In particular cases, these can be either real numbers, or only the numbers 0 or 1, or logical variables, propositions, predicates, etc.

Let everywhere below $\circ, * : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{X}$ be fixed operations and let $\langle \mathcal{X}, \circ, e_\circ \rangle$ and $\langle \mathcal{X}, *, e_* \rangle$ be groups (see, e.g., [63]) with unit elements $e_\circ$ and e_* , respectively. These notations were first proposed in [27], i.e., after the publication of [20]. We will mention that the elements of the group $\langle \mathcal{X}, \square, e_\square \rangle$ satisfy the following conditions (as the next equalities must be valid for both operations $\circ$ and $*$, below we will use notation $\square$ for each of these operations):

- $(\forall x, y \in \mathcal{X})(x \square y \in \mathcal{X}),$
- $(\forall x, y, z \in \mathcal{X})((x \square y) \square z = x \square (y \square z)),$
- $(\forall x \in \mathcal{X})(x \square e_\square = x = e_\square \square x),$
- $(\forall x \in \mathcal{X})(\exists y \in \mathcal{X})(x \square y = e_\square = y \square x).$

The 2-Dimensional IM (2DIM; but for brevity, only IM) with index sets K and L ($K, L \subset \mathcal{I}^*$) and elements from set $\mathcal{X}$ is called the object (see, [3, 20]):

$$[K, L, \{a_{k_i, l_j}\}] \equiv \begin{array}{c|ccccc} & l_1 & \dots & l_j & \dots & l_n \\ \hline k_1 & a_{k_1, l_1} & \dots & a_{k_1, l_j} & \dots & a_{k_1, l_n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_i & a_{k_i, l_1} & \dots & a_{k_i, l_j} & \dots & a_{k_i, l_n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_m & a_{k_m, l_1} & \dots & a_{k_m, l_j} & \dots & a_{k_m, l_n} \end{array},$$

where $K = \{k_1, k_2, \dots, k_m\}$, $L = \{l_1, l_2, \dots, l_n\}$, for $1 \leq i \leq m$, and $1 \leq j \leq n$: $a_{k_i, l_j} \in \mathcal{X}$. When $\mathcal{X}$ is:

- the set of real numbers $\mathbb{R}$, then the IM is over $\mathbb{R}$ and it is a $\mathbb{R}$ -IM,
- the set $\{0, 1\}$, then it is $\{0, 1\}$ -IM,
- the set of propositions or predicates, then it is a logical IM.

In the next chapters, we discuss other types of IM, as well.

1.2 Operations and relations over $\mathbb{R}$ -IMs and $(0, 1)$ -IMs

For the IMs

$$\begin{aligned} A &= [K, L, \{a_{k_i, l_j}\}], \\ B &= [P, Q, \{b_{p_r, q_s}\}], \end{aligned}$$

operations that are analogous to the usual matrix operations of addition and multiplication are defined, as well as other, specific ones.

Addition

$$A \oplus_{(\circ)} B = [K \cup P, L \cup Q, \{c_{t_u, v_w}\}],$$

where

$$c_{t_u, v_w} = \begin{cases} a_{k_i, l_j}, & \text{if } t_u = k_i \in K \text{ and } v_w = l_j \in L - Q \\ & \text{or } t_u = k_i \in K - P \text{ and } v_w = l_j \in L; \\ b_{p_r, q_s}, & \text{if } t_u = p_r \in P \text{ and } v_w = q_s \in Q - L \\ & \text{or } t_u = p_r \in P - K \text{ and } v_w = q_s \in Q; \\ a_{k_i, l_j} \circ b_{p_r, q_s}, & \text{if } t_u = k_i = p_r \in K \cap P \\ & \text{and } v_w = l_j = q_s \in L \cap Q; \\ 0, & \text{otherwise.} \end{cases}$$

Of course, here and below, if “ $\circ$ ” is substituted by “ $+$ ”, then $a_{k_i, l_j} \circ b_{p_r, q_s} = a_{k_i, l_j} + b_{p_r, q_s}$, while, if “ $\circ$ ” is “max” or min, then $a_{k_i, l_j} \circ b_{p_r, q_s} = \max(a_{k_i, l_j}, b_{p_r, q_s})$ or $a_{k_i, l_j} \circ b_{p_r, q_s} = \min(a_{k_i, l_j}, b_{p_r, q_s})$, respectively.

The visualization of operation $\oplus_{(\circ)}$ is given in Figure 1.1.

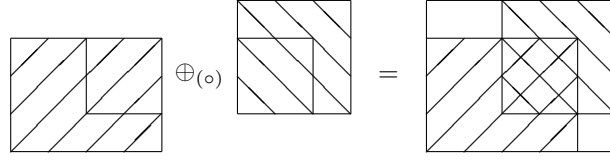


Figure 1.1. Geometrical interpretation of operation $\oplus_{(\circ)}$

Termwise multiplication

$$A \otimes_{(\circ)} B = [K \cap P, L \cap Q, \{c_{t_u, v_w}\}],$$

where $c_{t_u, v_w} = a_{k_i, l_j} \circ b_{p_r, q_s}$, for $t_u = k_i = p_r \in K \cap P$ and $v_w = l_j = q_s \in L \cap Q$.

The visualization of operation $\otimes_{(\circ)}$ is given in Figure 1.2.

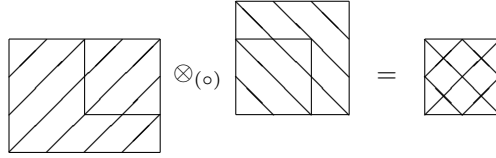


Figure 1.2. Geometrical interpretation of operation $\otimes_{(\circ)}$

Multiplication

$$A \odot_{(\circ, *)} B = [K \cup (P - L), Q \cup (L - P), \{c_{t_u, v_w}\}],$$

where

$$c_{t_u, v_w} = \begin{cases} a_{k_i, l_j}, & \text{if } t_u = k_i \in K \\ & \text{and } v_w = l_j \in L - P - Q; \\ b_{p_r, q_s}, & \text{if } t_u = p_r \in P - L - K \\ & \text{and } v_w = q_s \in Q; \\ \underset{l_j = p_r \in L \cap P}{\circ} (a_{k_i, l_j} * b_{p_r, q_s}), & \text{if } t_u = k_i \in K \text{ and } v_w = q_s \in Q; \\ 0, & \text{otherwise.} \end{cases}$$

Obviously, when $\circ$ is substituted by $+$, the symbol $\underset{j}{\circ}$ is substituted by $\sum_j$.

The visualization of the multiplication between two ordinary matrices is illustrated in Figure 1.3, while the visualization of the multiplication between two IMs is given in Figure 1.4.

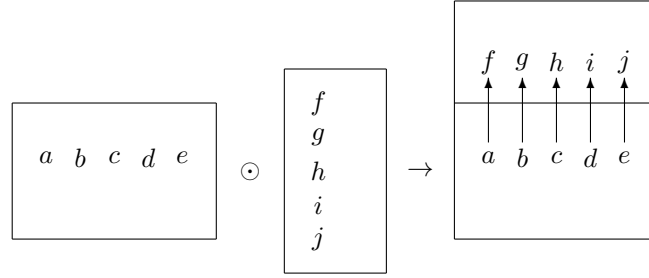


Figure 1.3. Geometrical interpretation of multiplication of two ordinary matrices

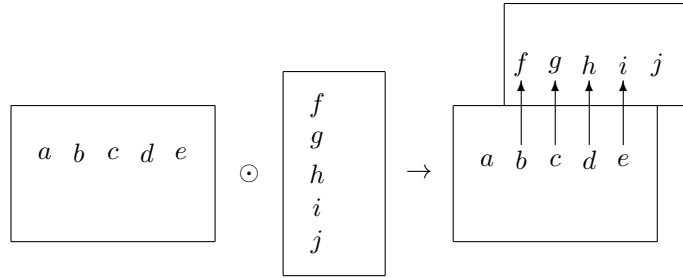


Figure 1.4. Geometrical interpretation of multiplication of two index matrices

We see immediately that

$$\begin{aligned} A^2 &= A \odot_{(\circ,*)} A = [K \cup (K - L), L \cup (L - K), \{c_{t_u, v_w}\}] \\ &= [K, L, \{c_{t_u, v_w}\}], \end{aligned}$$

where $c_{t_u, v_w} = \underset{l_j = p_r \in L \cap P}{\circ} a_{k_i, l_j} * b_{p_r, q_s}$ and for $n \geq 2$: $A^{n+1} = A^n \odot_{(\circ,*)} A$.

Structural subtraction

$$A \ominus B = [K - P, L - Q, \{c_{t_u, v_w}\}],$$

where “ $-$ ” is the set-theoretic operation difference and $c_{t_u, v_w} = a_{k_i, l_j}$, for $t_u = k_i \in K - P$ and $v_w = l_j \in L - Q$.

The visualization of operation $\ominus$ is shown in the following Figure 1.5.

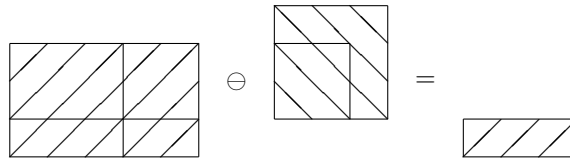


Figure 1.5. The geometrical interpretation of operation $\ominus$

Operation $\circ$ with a constant

In [20] this operation is called “Multiplication with a constant”, but here we extend it as follows. Let $\circ$ be some operation over the real numbers and let α be a real number. Then

$$\alpha \circ A = [K, L, \{\alpha \circ a_{k_i, l_j}\}].$$

In [20], operation $\circ$ is only multiplication. In the case of $(0,1)$ -IMs, $\circ \in \{\min, \max\}$. When the elements of the IM are propositions or predicates, then $\circ \in \{\vee, \wedge, \rightarrow, \dots\}$.

It is worth mentioning that for any two IMs A and B , for which $K \cap P = L \cap Q = \emptyset$,

$$A \oplus_{(+)} B = A \oplus_{(\times)} B = A \oplus_{(\max)} B = A \oplus_{(\min)} B.$$

For example, for IMs X and Y , and $a, b, c, d, e, p, q, r, s, t, u \in \mathcal{I}$, we have

$$X = \begin{array}{c|ccc} & c & d & e \\ a & 1 & 2 & 3 \\ b & 4 & 5 & 6 \end{array}, \quad Y = \begin{array}{c|cc} & c & r \\ a & 10 & 11 \\ p & 12 & 13 \\ q & 14 & 15 \end{array},$$

$$X \ominus Y = \begin{array}{c|cc} & d & e \\ b & 5 & 6 \end{array}, \quad X \oplus_{(-)} Y = \begin{array}{c|cccc} & c & d & e & r \\ a & -9 & 2 & 3 & -11 \\ b & 4 & 5 & 6 & 0 \\ p & -12 & 0 & 0 & -13 \\ q & -14 & 0 & 0 & -15 \end{array}.$$

If the IM Z has the form

$$Z = \begin{array}{c|c} & u \\ c & 10 \\ d & 11 \\ s & 12 \\ t & 13 \end{array},$$

then

$$X \odot_{(+, \times)} Z = \begin{array}{c|cc} & e & u \\ a & 3 & 1 \times 10 + 2 \times 11 \\ b & 6 & 4 \times 10 + 5 \times 11 \\ s & 0 & 12 \\ t & 0 & 13 \end{array} = \begin{array}{c|cc} & e & u \\ a & 3 & 32 \\ b & 6 & 95 \\ s & 0 & 12 \\ t & 0 & 13 \end{array}.$$

Since operation $\odot_{(\circ, *)}$ is one of the most complex, we illustrate it with two examples.

First, let us have the IM

$$A = \begin{array}{c|ccc} & d & e & f \\ \hline a & \alpha & \beta & \gamma \\ b & \delta & \varepsilon & \zeta \\ c & \eta & \theta & \iota \\ d & \kappa & \lambda & \mu \end{array},$$

where $\alpha, \beta, \dots, \mu \in \{0, 1\}$. Then

$$\begin{aligned} A^2 = A \odot_{(\max, \min)} A &= \begin{array}{c|ccc} & d & e & f \\ \hline a & \alpha & \beta & \gamma \\ b & \delta & \varepsilon & \zeta \\ c & \eta & \theta & \iota \\ d & \kappa & \lambda & \mu \end{array} \odot_{(\max, \min)} \begin{array}{c|ccc} & d & e & f \\ \hline a & \alpha & \beta & \gamma \\ b & \delta & \varepsilon & \zeta \\ c & \eta & \theta & \iota \\ d & \kappa & \lambda & \mu \end{array} \\ &= \begin{array}{c|ccc} & d & e & f \\ \hline a & \min(\alpha, \kappa) & \min(\alpha, \lambda) & \min(\alpha, \mu) \\ b & \min(\delta, \kappa) & \min(\delta, \lambda) & \min(\delta, \mu) \\ c & \min(\eta, \kappa) & \min(\eta, \lambda) & \min(\eta, \mu) \\ d & \min(\kappa, \kappa) & \min(\kappa, \lambda) & \min(\kappa, \mu) \end{array}. \end{aligned}$$

Indeed, the pair of indices (a, d) in the resultant matrix is met in the two IMs (that coincide) and, therefore, we cannot use the first two parts of formula for multiplication. In this case, we must use the third part of the same formula. Since $\{a, b, c, d\} \cap \{d, e, f\} = \{d\}$, the third operation will be reduced to the form $\min(x, y)$. Therefore, the element of the resultant IM, determined by the pair of indices (a, d) is exactly $\min(\alpha, \kappa)$. All other elements of the resultant IM are obtained in similar ways.

Now, let us have the IM

$$B = \begin{array}{c|ccc} & d & e & f \\ \hline a & \alpha & \beta & \gamma \\ b & \delta & \varepsilon & \zeta \\ e & \eta & \theta & \iota \\ d & \kappa & \lambda & \mu \end{array}.$$

where $\alpha, \beta, \dots, \mu \in \{0, 1\}$. Then we obtain sequentially¹

$$B^2 = B \odot_{(\min, \max)} B = \begin{array}{c|ccc} & d & e & f \\ \hline a & \alpha & \beta & \gamma \\ b & \delta & \varepsilon & \zeta \\ e & \eta & \theta & \iota \\ d & \kappa & \lambda & \mu \end{array} \odot_{(\min, \max)} \begin{array}{c|ccc} & d & e & f \\ \hline a & \alpha & \beta & \gamma \\ b & \delta & \varepsilon & \zeta \\ e & \eta & \theta & \iota \\ d & \kappa & \lambda & \mu \end{array}$$

¹ Here, in the IM we use symbol “...” in the end of the rows of the first IM and in the beginning of the second IM to denote that the IM is divided into two parts, due to the lack of place.

	d	e	$\dots$
a	$\min(\max(\alpha, \kappa), \max(\beta, \eta))$	$\min(\max(\alpha, \lambda), \max(\beta, \theta))$	$\dots$
$= b$	$\min(\max(\delta, \kappa), \max(\varepsilon, \eta))$	$\min(\max(\delta, \lambda), \max(\varepsilon, \theta))$	$\dots$
e	$\min(\max(\eta, \kappa), \max(\theta, \eta))$	$\min(\max(\eta, \lambda), \theta)$	$\dots$
d	$\min(\kappa, \max(\lambda, \eta))$	$\min(\max(\kappa, \lambda), \max(\varepsilon, \theta))$	$\dots$

	$\dots$	f
$\dots$	$\min(\max(\alpha, \mu), \max(\beta, \iota))$	
$\dots$	$\min(\max(\delta, \mu), \max(\varepsilon, \iota))$	$\dots$
$\dots$	$\min(\max(\eta, \mu), \max(\theta, \iota))$	
$\dots$	$\min(\max(\kappa, \mu), \max(\lambda, \iota))$	

Now, we can directly see that when $K = P = \{1, 2, \dots, m\}$ and $L = Q = \{1, 2, \dots, n\}$ we obtain the definitions for standard matrix operations.

In the IM case, we can use different symbols as indices of the rows and columns and they give us additional information and possibilities for description.

Let $\mathcal{IM}_{\mathbb{R}}$ be the set of all $\mathbb{R}$ -IMs and let

$$I_{\emptyset} = [\emptyset, \emptyset, \perp],$$

where symbol “ $\perp$ ” denotes the lack of IM-elements. Let, as above, $\circ, * \in \{+, \times, \max, \min\}$ and $(\circ, *) \in \{(+, \times), (\max, \min), (\min, \max)\}$.

The following assertions for the IM are discussed in [3].

- Theorem 1.** (a) $\langle \mathcal{IM}_{\mathbb{R}}, \oplus_{\circ} \rangle$ is a commutative semigroup,
 (b) $\langle \mathcal{IM}_{\mathbb{R}}, \otimes_{\circ} \rangle$ is a commutative semigroup,
 (c) $\langle \mathcal{IM}_{\mathbb{R}}, \odot_{\circ, +} \rangle$ is a semigroup,
 (d) $\langle \mathcal{IM}_{\mathbb{R}}, \oplus_{\circ}, I_{\emptyset} \rangle$ is a commutative monoid.

Let the IM A be given and let $k_0 \notin K$ and $l_0 \notin L$ be two indices. Now, following the paper [48] by Sotirova, Bureva and the author, we introduce the following eight aggregation operations over it:

Max-row-aggregation

$$\rho_{\max}(A, k_0) = k_0 \left| \begin{array}{cccc} l_1 & l_2 & \dots & l_n \\ \max_{1 \leq i \leq m} (a_{k_i, l_1}) & \max_{1 \leq i \leq m} (a_{k_i, l_2}) & \dots & \max_{1 \leq i \leq m} (a_{k_i, l_n}) \end{array} \right|,$$

Min-row-aggregation

$$\rho_{\min}(A, k_0) = k_0 \left| \begin{array}{cccc} l_1 & l_2 & \dots & l_n \\ \min_{1 \leq i \leq m} (a_{k_i, l_1}) & \min_{1 \leq i \leq m} (a_{k_i, l_2}) & \dots & \min_{1 \leq i \leq m} (a_{k_i, l_n}) \end{array} \right|,$$

Sum-row-aggregation

$$\rho_{\text{sum}}(A, k_0) = \frac{\begin{array}{c|cccc} & l_1 & l_2 & \dots & l_n \\ \hline k_0 & \sum_{i=1}^m a_{k_i, l_1} & \sum_{i=1}^m a_{k_i, l_2} & \dots & \sum_{i=1}^m a_{k_i, l_n} \end{array}}{k_0},$$

Average-row-aggregation

$$\rho_{\text{ave}}(A, k_0) = \frac{\begin{array}{c|cccc} & l_1 & l_2 & \dots & l_n \\ \hline k_0 & \frac{1}{m} \sum_{i=1}^m a_{k_i, l_1} & \frac{1}{m} \sum_{i=1}^m a_{k_i, l_2} & \dots & \frac{1}{m} \sum_{i=1}^m a_{k_i, l_n} \end{array}}{k_0},$$

Max-column-aggregation

$$\sigma_{\text{max}}(A, l_0) = \frac{\begin{array}{c|c} & l_0 \\ \hline k_1 & \max_{1 \leq j \leq n} a_{k_1, l_j} \\ \vdots & \vdots \\ k_m & \max_{1 \leq j \leq n} a_{k_m, l_j} \end{array}}{k_1},$$

Min-column-aggregation

$$\sigma_{\text{min}}(A, l_0) = \frac{\begin{array}{c|c} & l_0 \\ \hline k_1 & \min_{1 \leq j \leq n} a_{k_1, l_j} \\ \vdots & \vdots \\ k_m & \min_{1 \leq j \leq n} a_{k_m, l_j} \end{array}}{k_1},$$

Sum-column-aggregation

$$\sigma_{\text{sum}}(A, l_0) = \frac{\begin{array}{c|c} & l_0 \\ \hline k_1 & \sum_{j=1}^n a_{k_1, l_j} \\ \vdots & \vdots \\ k_m & \sum_{j=1}^n a_{k_m, l_j} \end{array}}{k_1},$$

Average-column-aggregation

$$\sigma_{\text{ave}}(A, l_0) = \frac{\begin{array}{c|c} & l_0 \\ \hline k_1 & \frac{1}{n} \sum_{j=1}^n a_{k_1, l_j} \\ \vdots & \vdots \\ k_m & \frac{1}{n} \sum_{j=1}^n a_{k_m, l_j} \end{array}}{k_1}.$$

We can see immediately that for every IM A , for every pair of indices i and j and for every $\circ \in \{\max, \min, \text{sum}, \text{ave}\}$:

- (1) $\rho_\circ(\rho_\circ(A, j), i) = \rho_\circ(A, i)$,
- (2) $\sigma_\circ(\sigma_\circ(A, i), j) = \sigma_\circ(A, j)$,
- (3) $\rho_\circ(\sigma_\circ(A, j), i) = \sigma_\circ(\rho_\circ(A, i), j)$.

In the case of $(0,1)$ -IMs, only operations Max-row-, Min-row-, Max-column- and Min-column aggregation are possible.

Let the two IMs $A = [K, L, \{a_{k,l}\}]$ and $B = [P, Q, \{b_{p,q}\}]$ be given. We shall introduce the following (new) definitions where $\subset$ and $\subseteq$ denote the relations “*strong inclusion*” and “*weak inclusion*”.

The strict relation “inclusion about dimension” is

$$A \subset_d B \text{ iff } (((K \subset P) \& (L \subset Q)) \vee ((K \subseteq P) \& (L \subset Q)) \vee ((K \subset P) \& (L \subseteq Q))) \& (\forall k \in K)(\forall l \in L)(a_{k,l} = b_{k,l}).$$

The non-strict relation “inclusion about dimension” is

$$A \subseteq_d B \text{ iff } (K \subseteq P) \& (L \subseteq Q) \& (\forall k \in K)(\forall l \in L)(a_{k,l} = b_{k,l}).$$

The strict relation “inclusion about value” is

$$A \subset_v B \text{ iff } (K = P) \& (L = Q) \& (\forall k \in K)(\forall l \in L)(a_{k,l} < b_{k,l}).$$

The non-strict relation “inclusion about value” is

$$A \subseteq_v B \text{ iff } (K = P) \& (L = Q) \& (\forall k \in K)(\forall l \in L)(a_{k,l} \leq b_{k,l}).$$

The strict relation “inclusion” is

$$A \subset_* B \text{ iff } (((K \subset P) \& (L \subset Q)) \vee ((K \subseteq P) \& (L \subset Q)) \vee ((K \subset P) \& (L \subseteq Q))) \& (\forall k \in K)(\forall l \in L)(a_{k,l} < b_{k,l}).$$

The non-strict relation “inclusion” is

$$A \subseteq_* B \text{ iff } (K \subseteq P) \& (L \subseteq Q) \& (\forall k \in K)(\forall l \in L)(a_{k,l} \leq b_{k,l}).$$

It can be directly seen that for every two IMs A and B ,

- if $A \subset_d B$, then $A \subseteq_d B$;
- if $A \subset_v B$, then $A \subseteq_v B$;
- if $A \subset B$, $A \subseteq_d B$, or $A \subseteq_v B$, then $A \subseteq B$;
- if $A \subset_d B$ or $A \subset_v B$, then $A \subseteq B$.

Similar properties are also valid for the relations discussed in the following chapters, and for this reason they are not explicitly mentioned.

1.3 IMs with elements logical variables, propositions or predicates

When we choose to work with matrices, whose elements are logical variables, propositions or predicates – let us call these IM “Logical IMs (L-IMs)”, the form of the IM from Section 1 is presented, but there are differences in the forms of the operations and relations.

The operations from Section 1.2 now obtain the following forms.

$$A \oplus_{(\circ)} B = [K \cup P, L \cup Q, \{c_{t_u, v_w}\}],$$

where

$$c_{t_u, v_w} = \begin{cases} a_{k_i, l_j}, & \text{if } t_u = k_i \in K \text{ and } v_w = l_j \in L - Q \\ & \text{or } t_u = k_i \in K - P \text{ and } v_w = l_j \in L; \\ b_{p_r, q_s}, & \text{if } t_u = p_r \in P \text{ and } v_w = q_s \in Q - L \\ & \text{or } t_u = p_r \in P - K \text{ and } v_w = q_s \in Q; \\ a_{k_i, l_j} \circ b_{p_r, q_s}, & \text{if } t_u = k_i = p_r \in K \cap P \\ & \text{and } v_w = l_j = q_s \in L \cap Q; \\ false, & \text{otherwise,} \end{cases}$$

where here and below in this section $\circ \in \{\wedge, \vee, \rightarrow, \equiv\}$.

$$A \otimes_{(\circ)} B = [K \cap P, L \cap Q, \{c_{t_u, v_w}\}],$$

where $c_{t_u, v_w} = a_{k_i, l_j} \circ b_{p_r, q_s}$, for $t_u = k_i = p_r \in K \cap P$ and $v_w = l_j = q_s \in L \cap Q$.

$$A \odot_{(\circ, *)} B = [K \cup (P - L), Q \cup (L - P), \{c_{t_u, v_w}\}],$$

where

$$c_{t_u, v_w} = \begin{cases} a_{k_i, l_j}, & \text{if } t_u = k_i \in K \\ & \text{and } v_w = l_j \in L - P - Q; \\ b_{p_r, q_s}, & \text{if } t_u = p_r \in P - L - K \\ & \text{and } v_w = q_s \in Q; \\ l_{j=p_r} \circ_{L \cap P} (a_{k_i, l_j} * b_{p_r, q_s}), & \text{if } t_u = k_i = p_r \in K \text{ and } v_w = q_s \in Q; \\ false, & \text{otherwise,} \end{cases}$$

where $(\circ, *) \in \{(\vee, \wedge), (\wedge, \vee)\}$.

Operation Structural subtraction from Section 1.2 preserves its form, while operation with a constant from the same section is impossible in general. It is possible only in the form

$$\neg A = [K, L, \{\neg a_{k_i, l_j}\}],$$

where “ $\neg$ ” means the operation “negation”. This notation is used, because of the variety of different possible forms of the operation “negation”, as discussed in Chapter 4.

Only operations Max-row-, Min-row-, Max-column- and Min-column aggregation from Section 1.2 exist for L-IMs, but in the following forms.

$\vee$ -row-aggregation

$$\rho_{\vee}(A, k_0) = \begin{array}{c|cccc} & l_1 & l_2 & \dots & l_n \\ \hline k_0 & \bigvee_{1 \leq i \leq m} (a_{k_i, l_1}) & \bigvee_{1 \leq i \leq m} (a_{k_i, l_2}) & \dots & \bigvee_{1 \leq i \leq m} (a_{k_i, l_n}) \end{array},$$

$\wedge$ -row-aggregation

$$\rho_{\wedge}(A, k_0) = \begin{array}{c|cccc} & l_1 & l_2 & \dots & l_n \\ \hline k_0 & \bigwedge_{1 \leq i \leq m} (a_{k_i, l_1}) & \bigwedge_{1 \leq i \leq m} (a_{k_i, l_2}) & \dots & \bigwedge_{1 \leq i \leq m} (a_{k_i, l_n}) \end{array},$$

$\vee$ -column-aggregation

$$\sigma_{\vee}(A, l_0) = \begin{array}{c|c} & l_0 \\ \hline k_1 & \bigvee_{1 \leq j \leq n} a_{k_1, l_j} \\ \vdots & \vdots \\ k_m & \bigvee_{1 \leq j \leq n} a_{k_m, l_j} \end{array},$$

$\wedge$ -column-aggregation

$$\sigma_{\wedge}(A, l_0) = \begin{array}{c|c} & l_0 \\ \hline k_1 & \bigwedge_{1 \leq j \leq n} a_{k_1, l_j} \\ \vdots & \vdots \\ k_m & \bigwedge_{1 \leq j \leq n} a_{k_m, l_j} \end{array},$$

In Chapter 2, other non-standard (for matrix theory) operations will be described that can be used for the IMs discussed in the present Chapter.

1.4 Examples from number theory

It is well-known (see, e.g., [65]) that each natural number n has a canonical representation $m = \prod_{i=1}^k p_i^{\alpha_i}$, where $k, \alpha_1, \alpha_2, \dots, \alpha_k \geq 1$ are natural numbers and $p_1, p_2, \dots, p_k$ are different prime numbers. We can always suppose that $p_1 < p_2 < \dots < p_k$.

As it is shown in [20], m has the following IM-interpretation:

$$IM(m, a) = \frac{p_1 \ p_2 \ \cdots \ p_k}{a \mid \alpha_1 \ \alpha_2 \ \cdots \ \alpha_k},$$

where “ a ” is an arbitrary symbol.

Obviously, if m is a prime number, its IM-interpretation is

$$IM(p, a) = \frac{p}{a \mid 1}.$$

Let us have two natural numbers m and n . In the general case, they have the forms

$$m = \prod_{i=1}^{k+h} p_i^{\alpha_i} \quad \text{and} \quad n = \prod_{i=k+l+1}^{k+h+g} p_i^{\beta_i}.$$

As it is shown in [20],

$$m.n = \left(\prod_{i=1}^k p_i^{\alpha_i} \right) \cdot \left(\prod_{i=k+1}^{k+h} p_i^{\alpha_i + \beta_i} \right) \cdot \left(\prod_{i=i+h+1}^{k+h+g} p_i^{\beta_i} \right).$$

Now, we see that

$$\begin{aligned} & IM(m, a) \oplus_{(+)} IM(n, a) \\ &= \frac{p_1 \ \cdots \ p_k \quad p_{k+1} \quad \cdots \quad p_{k+h} \quad p_{k+h+1} \ \cdots \ p_{k+h+g}}{a \mid \alpha_1 \ \cdots \ \alpha_k \ \alpha_{k+1} + \beta_{k+1} \ \cdots \ \alpha_{k+h} + \beta_{k+h} \ \beta_{k+h+1} \ \cdots \ \beta_{k+h+g}} \\ &= IM(m.n, a), \end{aligned}$$

and

$$IM(m^s, a) = n.IM(m, a) = \frac{p_1 \ p_2 \ \cdots \ p_k}{a \mid s\alpha_1 \ s\alpha_2 \ \cdots \ s\alpha_k},$$

for the natural number s , and

$$\begin{aligned} & IM(m, a) \oplus_{(\max)} IM(n, a) = \\ & \frac{p_1 \ \cdots \ p_k \quad p_{k+1} \quad \cdots \quad p_{k+h} \quad p_{k+h+1} \ \cdots \ p_{k+h+g}}{a \mid \alpha_1 \ \cdots \ \alpha_k \ \max(\alpha_{k+1}, \beta_{k+1}) \ \cdots \ \max(\alpha_{k+h}, \beta_{k+h}) \ \beta_{k+h+1} \ \cdots \ \beta_{k+h+g}}, \end{aligned}$$

that is an IM-interpretation of the least common multiple of m and n . The greatest common divisor of m and n has an IM-interpretation in the form

$$IM(m, a) \otimes_{(\min)} IM(n, a) = \frac{p_{k+1} \quad \cdots \quad p_{k+h}}{a \mid \min(\alpha_{k+1}, \beta_{k+1}) \ \cdots \ \min(\alpha_{k+h}, \beta_{k+h})}.$$

Finally, we will give examples with the following two new arithmetic operations, introduced by the author in [45]:

$$m *_1 n = \prod_{i=1}^k p_i^{\alpha_i} \prod_{i=k+l+1}^{k+l+r} p_i^{\beta_i} \quad (1)$$

and

$$m *_2 n = \prod_{i=1}^k p_i^{\alpha_i} \prod_{i=k+1}^{k+l} p_i^{\min(\alpha_i, \beta_i)} \prod_{i=k+l+1}^{k+l+r} p_i^{\beta_i}.$$

Then

$$\begin{aligned} & IM(m, a) \oplus_{(*_1)} IM(n, a) \\ &= \frac{a \left| \begin{array}{ccccccccc} p_1 & \cdots & p_k & p_{k+1} & \cdots & p_{k+h} & p_{k+h+1} & \cdots & p_{k+h+g} \\ \alpha_1 & \cdots & \alpha_k & 0 & \cdots & 0 & \beta_{k+h+1} & \cdots & \beta_{k+h+g} \end{array} \right|}{a} \\ &= IM(m *_1 n, a), \end{aligned}$$

and

$$\begin{aligned} & IM(m, a) \oplus_{(*_2)} IM(n, a) \\ &= \frac{a \left| \begin{array}{ccccccccc} p_1 & \cdots & p_k & p_{k+1} & \cdots & p_{k+h} & p_{k+h+1} & \cdots & p_{k+h+g} \\ \alpha_1 & \cdots & \alpha_k & \min(\alpha_{k+1}, \beta_{k+1}) & \cdots & \min(\alpha_{k+h}, \beta_{k+h}) & \beta_{k+h+1} & \cdots & \beta_{k+h+g} \end{array} \right|}{a} \\ &= IM(m *_2 n, a). \end{aligned}$$

1.5 Transposed IM

Let the IM A be given as above. Then its transposed IM has the form

$$A' = [L, K, \{a_{l_j, k_i}\}] = \begin{array}{c|cccc} & k_1 & \cdots & k_i & \cdots & k_m \\ \hline l_1 & a_{l_1, k_1} & \cdots & a_{l_1, k_i} & \cdots & a_{l_1, k_m} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ l_j & a_{l_j, k_1} & \cdots & a_{l_j, k_i} & \cdots & a_{l_j, k_m} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ l_n & a_{l_n, k_1} & \cdots & a_{l_n, k_i} & \cdots & a_{l_n, k_m} \end{array}.$$

The visualization of this operation is

$$\left(K \left| \begin{array}{c} L \\ \bullet \circ \end{array} \right. \right)' = L \left| \begin{array}{c} K \\ \bullet \\ \circ \end{array} \right.$$

The IMs $A \odot_{(\circ,*)} A'$ and $A' \odot_{(\circ,*)} A$ are square IMs. For example, if

$$A = \begin{array}{c|cc} & d & e \\ \hline a & \alpha & \beta \\ b & \gamma & \delta \\ c & \varepsilon & \zeta \end{array},$$

then

$$A' = \begin{array}{c|ccc} & a & b & c \\ \hline d & \alpha & \gamma & \varepsilon \\ e & \beta & \delta & \zeta \end{array}$$

and

$$A \odot_{(+,\times)} A' = \begin{array}{c|ccc} & a & b & c \\ \hline a & \alpha^2 + \beta^2 & \alpha\gamma + \beta\delta & \alpha\varepsilon + \beta\zeta \\ b & \alpha\gamma + \beta\delta & \gamma^2 & \gamma\varepsilon + \delta\zeta \\ c & \alpha\varepsilon + \beta\zeta & \gamma\varepsilon + \delta\zeta & \varepsilon^2 + \zeta^2 \end{array},$$

while

$$A' \odot_{(+,\times)} A = \begin{array}{c|ccc} & d & e & \\ \hline d & \alpha^2 + \gamma^2 + \varepsilon^2 & \alpha\beta + \gamma\delta + \varepsilon\zeta & \\ e & \alpha\beta + \gamma\delta + \varepsilon\zeta & \beta^2 + \delta^2 + \zeta^2 & \end{array}.$$

Therefore, in general case operation $\odot_{(\circ,*)}$ is not commutative.

1.6 Level operators over IMs

Let $\mathcal{I}$ be a fixed set of indices and φ, ψ be fixed real numbers, such that $\varphi < \psi$.

Following and extending the results from [32], let us have the IM $A = [K, L, \{a_{k_i, l_j}\}]$, where $a_{k_i, l_j} \in [\varphi, \psi]$. Then we define

$$\mathcal{L}_\alpha^>(A) = [K, L, \{b_{k_i, l_j}\}], \quad \text{where } b_{k_i, l_j} = \begin{cases} a_{k_i, l_j}, & \text{if } a_{k_i, l_j} > \alpha, \\ \varphi, & \text{otherwise;} \end{cases}$$

$$\mathcal{L}_\alpha^\geq(A) = [K, L, \{b_{k_i, l_j}\}], \quad \text{where } b_{k_i, l_j} = \begin{cases} a_{k_i, l_j}, & \text{if } a_{k_i, l_j} \geq \alpha, \\ \varphi, & \text{otherwise;} \end{cases}$$

$$\mathcal{L}_\alpha^<(A) = [K, L, \{b_{k_i, l_j}\}], \quad \text{where } b_{k_i, l_j} = \begin{cases} a_{k_i, l_j}, & \text{if } a_{k_i, l_j} < \alpha, \\ \psi, & \text{otherwise;} \end{cases}$$

$$\mathcal{L}_\alpha^\leq(A) = [K, L, \{b_{k_i, l_j}\}], \quad \text{where } b_{k_i, l_j} = \begin{cases} a_{k_i, l_j}, & \text{if } a_{k_i, l_j} \leq \alpha, \\ \psi, & \text{otherwise;} \end{cases}$$

Let for $K, L \subset \mathcal{I}$:

$$\Phi_{K,L} = [K, L, \{\varphi\}], \quad \Psi_{K,L} = [K, L, \{\psi\}].$$

Then for $\alpha \in [\varphi, \psi]$:

$$\mathcal{L}_\alpha^>(\Phi_{K,L}) = \Phi_{K,L},$$

$$\mathcal{L}_\alpha^\geq(\Phi_{K,L}) = \Phi_{K,L},$$

$$\mathcal{L}_\alpha^<(\Psi_{K,L}) = \Psi_{K,L},$$

$$\mathcal{L}_\alpha^\leq(\Psi_{K,L}) = \Psi_{K,L}.$$

For the four operators, we can prove the following assertions, a part of which are published in [32]:

- for each IM A and for every two numbers $\alpha, \beta \in [\varphi, \psi]$:

$$\mathcal{L}_\alpha^>(\mathcal{L}_\beta^>(A)) = \mathcal{L}_{\max(\alpha, \beta)}^>(A),$$

$$\mathcal{L}_\alpha^\geq(\mathcal{L}_\beta^\geq(A)) = \mathcal{L}_{\max(\alpha, \beta)}^\geq(A),$$

$$\mathcal{L}_\alpha^<(\mathcal{L}_\beta^<(A)) = \mathcal{L}_{\min(\alpha, \beta)}^<(A),$$

$$\mathcal{L}_\alpha^\leq(\mathcal{L}_\beta^\leq(A)) = \mathcal{L}_{\min(\alpha, \beta)}^\leq(A);$$

- for two IMs A and B and for an arbitrary real number $\alpha > 0$:

$$(a) \mathcal{L}_\alpha^>(A) \oplus_{(\max)} \mathcal{L}_\alpha^>(B) =_v \mathcal{L}_\alpha^>(A \oplus_{(\max)} B),$$

$$(b) \mathcal{L}_\alpha^>(A) \oplus_{(\min)} \mathcal{L}_\alpha^>(B) =_v \mathcal{L}_\alpha^>(A \oplus_{(\min)} B),$$

$$(c) \mathcal{L}_\alpha^\geq(A) \oplus_{(\max)} \mathcal{L}_\alpha^\geq(B) =_v \mathcal{L}_\alpha^\geq(A \oplus_{(\max)} B),$$

$$(d) \mathcal{L}_\alpha^\geq(A) \oplus_{(\min)} \mathcal{L}_\alpha^\geq(B) =_v \mathcal{L}_\alpha^\geq(A \oplus_{(\min)} B),$$

$$(e) \mathcal{L}_\alpha^<(A) \oplus_{(\max)} \mathcal{L}_\alpha^<(B) =_v \mathcal{L}_\alpha^<(A \oplus_{(\max)} B),$$

$$(f) \mathcal{L}_\alpha^<(A) \oplus_{(\min)} \mathcal{L}_\alpha^<(B) =_v \mathcal{L}_\alpha^<(A \oplus_{(\min)} B),$$

$$(g) \mathcal{L}_\alpha^\leq(A) \oplus_{(\max)} \mathcal{L}_\alpha^\leq(B) =_v \mathcal{L}_\alpha^\leq(A \oplus_{(\max)} B),$$

$$(h) \mathcal{L}_\alpha^\leq(A) \oplus_{(\min)} \mathcal{L}_\alpha^\leq(B) =_v \mathcal{L}_\alpha^\leq(A \oplus_{(\min)} B),$$

- for two IMs A and B , and for $\alpha \in [\varphi, \psi]$:

$$(a) \mathcal{L}_\alpha^>(A) \otimes_{(\max)} \mathcal{L}_\alpha^>(B) =_v \mathcal{L}_\alpha^>(A \otimes_{(\max)} B),$$

$$(b) \mathcal{L}_\alpha^>(A) \otimes_{(\min)} \mathcal{L}_\alpha^>(B) =_v \mathcal{L}_\alpha^>(A \otimes_{(\min)} B),$$

$$(c) \mathcal{L}_\alpha^\geq(A) \otimes_{(\max)} \mathcal{L}_\alpha^\geq(B) =_v \mathcal{L}_\alpha^\geq(A \otimes_{(\max)} B),$$

$$(d) \mathcal{L}_\alpha^\geq(A) \otimes_{(\min)} \mathcal{L}_\alpha^\geq(B) =_v \mathcal{L}_\alpha^\geq(A \otimes_{(\min)} B),$$

$$(e) \mathcal{L}_\alpha^<(A) \otimes_{(\max)} \mathcal{L}_\alpha^<(B) =_v \mathcal{L}_\alpha^<(A \otimes_{(\max)} B),$$

$$(f) \mathcal{L}_\alpha^<(A) \otimes_{(\min)} \mathcal{L}_\alpha^<(B) =_v \mathcal{L}_\alpha^<(A \otimes_{(\min)} B),$$

$$(g) \mathcal{L}_\alpha^\leq(A) \otimes_{(\max)} \mathcal{L}_\alpha^\leq(B) =_v \mathcal{L}_\alpha^\leq(A \otimes_{(\max)} B),$$

$$(h) \mathcal{L}_\alpha^\leq(A) \otimes_{(\min)} \mathcal{L}_\alpha^\leq(B) =_v \mathcal{L}_\alpha^\leq(A \otimes_{(\min)} B).$$

Extended IMs (EIMs)

The discussed above IMs are, in some sense, extensions of ordinary matrices. Now, following and extending the results from [20], we introduce an IM, that includes all of the IMs from Chapter 1 as particular cases.

2.1 Definition of an EIM

Let $\mathcal{I}$ be a fixed set of indices as in Section 1.1, and using it we can again construct sets $\mathcal{I}^k$ and $\mathcal{I}^*$ for an arbitrary natural number k .

Let operations $\circ, * : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{X}$ be fixed.

An Extended IM (EIM) with index sets K and L ($K, L \subset \mathcal{I}^*$) and elements from the arbitrary set $\mathcal{X}$ (see, section 1.1) is called the object (see, [20, 22]):

$$[K, L, \{a_{k_i, l_j}\}] \equiv \begin{array}{c|ccccc} & l_1 & \cdots & l_j & \cdots & l_n \\ \hline k_1 & a_{k_1, l_1} & \cdots & a_{k_1, l_j} & \cdots & a_{k_1, l_n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_i & a_{k_i, l_1} & \cdots & a_{k_i, l_j} & \cdots & a_{k_i, l_n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_m & a_{k_m, l_1} & \cdots & a_{k_m, l_j} & \cdots & a_{k_m, l_n} \end{array},$$

where $K = \{k_1, k_2, \dots, k_m\}$, $L = \{l_1, l_2, \dots, l_n\}$, for $1 \leq i \leq m$, and $1 \leq j \leq n$: $a_{k_i, l_j} \in \mathcal{X}$.

Unlike the set $\mathcal{X}$ and the IMs from Chapter 1, here the set $\mathcal{X}$ and the EIMs can contain arbitrary elements and in particular, whole IMs.

2.2 Standard operations over EIMs

Let in this section, the sets $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be fixed. Let operations “ $*$ ” and “ $\circ$ ” be defined so that $* : \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{Z}$ and $\circ : \mathcal{Z} \times \mathcal{Z} \rightarrow \mathcal{Z}$.

The operations from Section 1.2 remain valid here without changes.

Now, we see that for operations “addition” and “termwise multiplication”,

- in the case of standard IMs, i.e., $\mathbb{R}$ -IM, $\mathcal{X} = \mathcal{Y} = \mathbb{R}$, where here and below, $\mathbb{R}$ is the set of the real numbers, operation “*” is the standard operation “+” or “ $\times$ ” and, obviously, $\mathcal{Z} = \mathbb{R}$;
- when $\mathcal{X} = \mathcal{Y} = \{0, 1\}$, operation “*” is “max” or “min”, and $\mathcal{Z} = \mathcal{X}$;
- when $\mathcal{X} = \mathcal{Y}$ is a set of logical variables, sentences or predicates, then “*” is “ $\vee$ ” or “ $\wedge$ ” and $\mathcal{Z} = \mathcal{X}$;
- when

$$\mathcal{X} = \mathcal{Y} = \mathcal{L}^* \equiv \{\langle a, b \rangle \mid a, b, a + b \in [0, 1]\},$$

then $\mathcal{Z} = \mathcal{X}$ and operation “*” is defined for the Intuitionistic Fuzzy Pairs (IFPs, i.e., pairs with the form $\langle a, b \rangle$, where $a, b \in [0, 1]$ and $a + b \in \mathcal{L}^*$) $\langle a, b \rangle$ and $\langle c, d \rangle$, by

$$\langle a, b \rangle * \langle c, d \rangle = \langle \max(a, c), \min(b, d) \rangle$$

or

$$\langle a, b \rangle * \langle c, d \rangle = \langle \min(a, c), \max(b, d) \rangle.$$

The latest case will be discussed in more details in Chapter 4.

In the case of operation “multiplication”,

- in the case of standard IM, $\mathcal{X} = \mathcal{Y} = \mathbb{R}$, operation “*” is the standard operation “ $\times$ ” and operation “o” is the standard operation “ $\cdot$ ”, obviously, $\mathcal{Z} = \mathbb{R}$;
- when $\mathcal{X} = \mathcal{Y} = \{0, 1\}$, operation “*” is “max” and “o” – “min”, or oppositely, “*” is “min” and “o” – “max”, and $\mathcal{Z} = \mathcal{X}$;
- when $\mathcal{X} = \mathcal{Y}$ are a set of logical variables, sentences or predicates, then “*” is “ $\vee$ ” and “o” – “ $\wedge$ ”, or vice versa, “*” is “ $\wedge$ ” and “o” – “ $\vee$ ”, and $\mathcal{Z} = \mathcal{X}$;
- when $\mathcal{X} = \mathcal{Y} = \mathcal{L}^*$, then $\mathcal{Z} = \mathcal{X}$ and operation * is defined for the intuitionistic fuzzy pairs $\langle a, b \rangle$ and $\langle c, d \rangle$, as above.

In the case of “operation with a constant”,

- if $\mathcal{X} = \mathbb{R}$, then the constant $\alpha \in \mathbb{R}$ and the operation $\circ \in \{+, -, \cdot, : \}$;
- if $\mathcal{X} = \{0, 1\}$, then $\alpha \in \{0, 1\}$ and the operation $\circ \in \{\max, \min\}$;
- when $\mathcal{X} = \mathcal{Y}$ is a set of logical variables, propositions or predicates, then α can be treat as the operation “negation”, but also as a variable, and in this case the operation $\circ \in \{\wedge, \vee, \rightarrow, \equiv, \dots\}$.
- when the set $\mathcal{X}$ contains IFPs, then the operation “*” is “max” and “o” is “min”, or vice versa, “*” is “min” and “o” is “max” and $\mathcal{X} = \mathcal{Y} = \mathcal{Z}$. In this case, as we will discuss in Chapter 4, the form of the indices can be different: now, it can be a triple consisting of an index and two numbers in the interval $[0, 1]$, representing the degrees of its validity (existence, etc.) and of its non-validity (non-existence, etc.).

2.3 Non-standard operations over EIMs

If $\circ : \mathcal{X} \times \dots \times \mathcal{X} \rightarrow \mathcal{X}$, where $\mathcal{X}$ is the set of natural ($\mathbb{N}$), real ($\mathbb{R}$) or complex ($\mathbb{C}$) numbers, or the set $\{0, 1\}$, then the aggregation operations have the forms from Chapter 1 and [78]. If the elements of the set $\mathcal{X}$ are arbitrary objects evaluable by some function Φ , then the aggregation operations have the forms:

$\circ$ -row-aggregation

$$\rho_{\circ}(A, k_0) = k_0 \left| \begin{array}{c|cccc} & l_1 & l_2 & \dots & l_n \\ \hline & \bigcirc_{i=1}^m \Phi(a_{k_i, l_1}) & \bigcirc_{i=1}^m \Phi(a_{k_i, l_2}) & \dots & \bigcirc_{i=1}^m \Phi(a_{k_i, l_n}) \end{array} \right|,$$

$\circ$ -column-aggregation

$$\sigma_{\circ}(A, l_0) = \left| \begin{array}{c|c} & l_0 \\ \hline k_1 & \bigcirc_{j=1}^n \Phi(a_{k_1, l_j}) \\ k_2 & \bigcirc_{j=1}^n \Phi(a_{k_2, l_j}) \\ \vdots & \vdots \\ k_m & \bigcirc_{j=1}^n \Phi(a_{k_m, l_j}) \end{array} \right|,$$

where, as explained above, when $\circ$ stays, for example, for the operation “+”, the “big” operator denoted by $\bigcirc_{j=1}^m$ will coincide with $\sum_{j=1}^m$.

Here and below we use symbol “ $\perp$ ” to indicate the lack of some component in the separate definitions. In some cases, when we use real numbers, it is suitable to change this symbol with “0”.

Operation “reduction” over an EIM

First, we introduce operations $(k, \perp)$ - and $(\perp, l)$ -reduction of a given IM $A = [K, L, \{a_{k_i, l_j}\}]$:

$$A_{(k, \perp)} = [K - \{k\}, L, \{c_{t_u, v_w}\}]$$

where $c_{t_u, v_w} = a_{k_i, l_j}$ for $t_u = k_i \in K - \{k\}$ and $v_w = l_j \in L$ and

$$A_{(\perp, l)} = [K, L - \{l\}, \{c_{t_u, v_w}\}],$$

where $c_{t_u, v_w} = a_{k_i, l_j}$ for $t_u = k_i \in K$ and $v_w = l_j \in L - \{l\}$.

Second, we define

$$A_{(k,l)} = (A_{(k,\perp)})_{(\perp,l)} = (A_{(\perp,l)})_{(k,\perp)},$$

i.e.,

$$A_{(k,l)} = [K - \{k\}, L - \{l\}, \{c_{t_u, v_w}\}],$$

where $c_{t_u, v_w} = a_{k_i, l_j}$ for $t_u = k_i \in K - \{k\}$ and $v_w = l_j \in L - \{l\}$.

We can check that for every IM A and for every $k_1, k_2 \in K$, $l_1, l_2 \in L$,

$$(A_{(k_1, l_1)})_{(k_2, l_2)} = (A_{(k_2, l_2)})_{(k_1, l_1)}.$$

Third, let $P = \{k_1, k_2, \dots, k_s\} \subseteq K$ and $Q = \{q_1, q_2, \dots, q_t\} \subseteq L$. Then, we define the following three operations:

$$A_{(P,l)} = (\dots((A_{(k_1,l)})_{(k_2,l)})\dots)_{(k_s,l)},$$

$$A_{(k,Q)} = (\dots((A_{(k,l_1)})_{(k,l_2)})\dots)_{(k,l_t)},$$

$$A_{(P,Q)} = (\dots((A_{(p_1,Q)})_{(p_2,Q)})\dots)_{(p_s,Q)} = (\dots((A_{(P,q_1)})_{(P,q_2)})\dots)_{(P,q_t)}.$$

Obviously,

$$A_{(K,L)} = I_\emptyset \quad \text{and} \quad A_{(\emptyset, \emptyset)} = A.$$

We can check that for every two IMs $A = [K, L, \{a_{k_i, l_j}\}]$, $B = [P, Q, \{b_{p_r, q_s}\}]$:

$$A \subseteq_d B \quad \text{iff} \quad A = B_{(P-K, Q-L)}.$$

Operation “projection” over an EIM

Let $M \subseteq K$ and $N \subseteq L$. Then,

$$\text{pr}_{M,N} A = [M, N, \{b_{k_i, l_j}\}],$$

where for each $k_i \in M$ and each $l_j \in N$, $b_{k_i, l_j} = a_{k_i, l_j}$.

Obviously, for every IM A and for the sets $M_1 \subseteq M_2 \subseteq K$ and $N_1 \subseteq N_2 \subseteq L$ the equality

$$\text{pr}_{M_1, N_1} \text{pr}_{M_2, N_2} A = \text{pr}_{M_1, N_1} A$$

holds.

We can check that for $M \subseteq K$, $N \subseteq L$:

$$\text{pr}_{M,N} A = A_{(K-M, L-N)},$$

$$A_{M,N} = \text{pr}_{K-M, L-N} A.$$

Operation “substitution” over an EIM

Let the IM $A = [K, L, \{a_{k,l}\}]$ be given.

First, local substitution over the IM is defined for the couples of indices (p, k) and/or (q, l) , respectively, by

$$\begin{aligned} \left[\frac{p}{k}; \perp\right] A &= [(K - \{k\}) \cup \{p\}, L, \{a_{k,l}\}], \\ \left[\perp; \frac{q}{l}\right] A &= [K, (L - \{l\}) \cup \{q\}, \{a_{k,l}\}], \end{aligned}$$

Second,

$$\left[\frac{p}{k}; \frac{q}{l}\right] A = \left[\frac{p}{k}; \perp\right] \left[\perp; \frac{q}{l}\right] A,$$

i.e.,

$$\left[\frac{p}{k}; \frac{q}{l}\right] A = [(K - \{k\}) \cup \{p\}, (L - \{l\}) \cup \{q\}, \{a_{k,l}\}].$$

Obviously, for the above indices k, l, p, q :

$$\begin{aligned} \left[\frac{k}{p}; \perp\right] \left(\left[\perp; \frac{p}{k}\right] A\right) &= \left[\frac{l}{q}; \perp\right] \left(\left[\perp; \frac{q}{l}\right] A\right), \\ \left[\frac{k}{p}; \frac{l}{q}\right] \left(\left[\frac{p}{k}; \frac{q}{l}\right] A\right) &= A. \end{aligned}$$

Third, let the sets of indices $P = \{p_1, p_2, \dots, p_u\}$, $Q = \{q_1, q_2, \dots, q_v\}$ be given. For them we define sequentially:

$$\begin{aligned} \left[\frac{P}{K}; \perp\right] A &= \left[\frac{p_1}{k_1} \frac{p_2}{k_2} \dots \frac{p_u}{k_u}; \perp\right] A, \\ \left[\perp; \frac{Q}{L}\right] A &= \left(\left[\perp; \frac{q_1}{l_1} \frac{q_2}{l_2} \dots \frac{q_v}{l_v}\right] A\right), \\ \left[\frac{P}{K}; \frac{Q}{L}\right] A &= \left[\frac{P}{K}; \perp\right] \left(\left[\perp; \frac{Q}{L}\right] A\right). \end{aligned}$$

Obviously, for the sets K, L, P, Q :

$$\left[\frac{K}{P}; \perp\right] \left(\left[\frac{P}{K}; \perp\right] A\right) = \left[\perp; \frac{L}{Q}\right] \left(\left[\perp; \frac{Q}{L}\right] A\right) = \left[\frac{K}{P}; \frac{L}{Q}\right] \left(\left[\frac{P}{K}; \frac{Q}{L}\right] A\right) = A.$$

We can check that for every four sets of indices P_1, P_2, Q_1, Q_2

$$\left[\frac{P_2}{P_1}; \frac{Q_2}{Q_1}\right] \left[\frac{P_1}{K}; \frac{Q_1}{L}\right] A = \left[\frac{P_2}{K}; \frac{Q_2}{L}\right] A.$$

Fourth, let for brevity $1 \leq i < j \leq n$. Then the local substitution of an element of A is defined by

$$\left[\left\langle \frac{b_{k_i, l_j}}{a_{k_i, l_j}} \right\rangle \right] \begin{array}{c|ccc} & \dots & l_j & \dots \\ \hline \vdots & \ddots & \vdots & \ddots \\ k_i & \dots & a_{k_i, l_j} & \dots \\ \vdots & \ddots & \vdots & \ddots \end{array} = \begin{array}{c|ccc} & \dots & l_j & \dots \\ \hline \vdots & \ddots & \vdots & \ddots \\ k_i & \dots & b_{k_i, l_j} & \dots \\ \vdots & \ddots & \vdots & \ddots \end{array}$$

The following operation are defined for index sets $K \subset P \subset \mathcal{I}$ and $L \subset Q \subset \mathcal{I}$ as follows:

- “**Inflating operation**”

$${}^{(P,Q)}A = {}^{(P,Q)}[K, L, \{a_{k_i, l_j}\}] = [P, Q, \{b_{p_r, q_s}\}],$$

where

$$b_{p_r, q_s} = \begin{cases} a_{k_i, l_j}, & \text{if } p_r = k_i \in K \text{ and } q_s = l_j \in L, \\ \perp, & \text{otherwise.} \end{cases}$$

- **Operation “Automatic reduction”**

$$@A = [P, Q, \{b_{p_r, q_s}\}],$$

where $P \subseteq K, Q \subseteq L$ are index sets with the following property:

$$(\forall k \in K - P)(\forall l \in L)(a_{k_i, l_j} = \perp) \ \& \ (\forall k \in K)(\forall l \in L - Q)(a_{k_i, l_j} = \perp) \\ \& \ (\forall p_r = k_i \in P)(\forall q_s = l_j \in Q)(b_{p_r, q_s} = a_{k_i, l_j}).$$

For example,

$$\text{if } A = \begin{array}{c|ccc} & d & e & f & g \\ \hline a & 1 & 2 & \perp & 3 \\ b & \perp & \perp & \perp & \perp \\ c & 4 & 5 & \perp & \perp \end{array}, \quad \text{then } @A = \begin{array}{c|ccc} & d & e & g \\ \hline a & 1 & 2 & 3 \\ c & 4 & 5 & \perp \end{array}.$$

All above operations are defined over a given EIM. Below, following, correcting and extending some results from [20], we introduce two (non-standard) operations over more than one EIMs.

The next operation is **operation “composition” over EIMs** is defined for n EIMs $A_1, A_2, \dots, A_n$. Let for s ($1 \leq s \leq n$):

$$(\forall p, q)(1 \leq p < q \leq n)(K^p \cap K^q = L^p \cap L^q = \emptyset)$$

and

$$A_s = [K^s, L^s, \{a_{k_i, l_j}^s\}] = \begin{array}{c|cccc} & l_{s,1} & \dots & l_{s,j} & \dots & l_{s,n_s} \\ \hline k_{s,1} & a_{k_{s,1}, l_{s,1}} & \dots & a_{k_{s,1}, l_{s,j}} & \dots & a_{k_{s,1}, l_{s,n_s}} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_{s,i} & a_{k_{s,i}, l_{s,1}} & \dots & a_{k_{s,i}, l_{s,j}} & \dots & a_{k_{s,i}, l_{s,n_s}} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_{s,m} & a_{k_{s,m}, l_{s,1}} & \dots & a_{k_{s,m}, l_{s,j}} & \dots & a_{k_{s,m}, l_{s,n_s}} \end{array}.$$

The operation $\flat$ is defined as:

$$\flat\{A_s | 1 \leq s \leq n\} = [\bigcup_{s=1}^n K^s, \bigcup_{s=1}^n L^s, \{\langle c_{1,t_1,u,v_1,w}, c_{2,t_2,u,v_2,w}, \dots, c_{n,t_n,u,v_n,w} \rangle\}],$$

where for r ($1 \leq r \leq n$):

$$c_{r,t_u,v_w} = \begin{cases} a_{r,k_i,l_j}, & \text{if } t_u = k_i \in K^r \text{ and } v_w = l_j \in L^r, \\ \perp, & \text{otherwise.} \end{cases}$$

Therefore, this operation is composed of a new EIM on the basis of n EIMs. The new EIM contains n -dimensional vectors as elements. By this reason, we define function $\dim$, giving the dimensionality of the elements of the EIM A , i.e., for the above EIM, the equality $\dim(A) = n$ holds. A modification of this operator is described in [16].

Let $\mathcal{X}$ be a set of n -dimensional vectors and

$$A = [K, L, \{\langle a_{k_i,l_j}^1, a_{k_i,l_j}^2, \dots, a_{k_i,l_j}^n \rangle\}]$$

be an EIM with elements from the set $\mathcal{X}$. Then for the natural number $s \geq 1$:

$$\Pr_s(A) = \begin{cases} I_\emptyset, & \text{if } s > n, \\ A_s, & \text{otherwise,} \end{cases},$$

where

$$A_s = [K, L, \{a_{k_i,l_j}^s\}]$$

and a_{k_i,l_j}^s is the s -th component of vector $\langle a_{k_i,l_j}^1, a_{k_i,l_j}^2, \dots, a_{k_i,l_j}^n \rangle$ that is an element of A .

Now, we can define an operation, that is in some sense opposite of the operation $\flat$. It has the form

$$\sharp(A) = \{\textcircled{\Pr}_s(A) | 1 \leq s \leq n\}.$$

Let us give an example. Let the EIMs A_1, A_2 have the forms

$$A_1 = \begin{array}{c|cccc} & d & e & f & g \\ \hline a & 1 & 2 & \perp & 3 \\ b & 4 & \perp & 5 & \perp \\ c & 6 & 7 & \perp & 8 \end{array}, \quad A_2 = \begin{array}{c|ccc} & d & i & f \\ \hline a & 11 & \perp & 12 \\ c & \perp & 13 & 14 \\ h & 15 & \perp & \perp \end{array}.$$

Then

$$A = \flat\{A_1, A_2\} = \begin{array}{c|ccccc} & d & e & f & g & i \\ \hline a & \langle 1, 11 \rangle & \langle 2, \perp \rangle & \langle \perp, 12 \rangle & \langle 3, \perp \rangle & \langle \perp, \perp \rangle \\ b & \langle 4, \perp \rangle & \langle \perp, \perp \rangle & \langle 5, \perp \rangle & \langle \perp, \perp \rangle & \langle \perp, \perp \rangle \\ c & \langle 6, \perp \rangle & \langle 7, \perp \rangle & \langle \perp, 14 \rangle & \langle 8, \perp \rangle & \langle \perp, 13 \rangle \\ h & \langle \perp, 15 \rangle & \langle \perp, \perp \rangle & \langle \perp, \perp \rangle & \langle \perp, \perp \rangle & \langle \perp, \perp \rangle \end{array}.$$

On the other hand,

$$\begin{aligned} \sharp(A) &= \{\textcircled{A}\text{Pr}_1(A), \textcircled{A}\text{Pr}_2(A)\} \\ &= \left\{ \textcircled{A} \left(\begin{array}{c|ccccc} & d & e & f & g & i \\ a & 1 & 2 & \perp & 3 & \perp \\ b & 4 & \perp & 5 & \perp & \perp \\ c & 6 & 7 & \perp & 8 & \perp \\ h & \perp & \perp & \perp & \perp & \perp \end{array} \right), \textcircled{A} \left(\begin{array}{c|ccccc} & d & e & f & g & i \\ a & 11 & \perp & 12 & \perp & \perp \\ b & \perp & \perp & \perp & \perp & \perp \\ c & \perp & \perp & 14 & \perp & 13 \\ h & 15 & \perp & \perp & \perp & \perp \end{array} \right) \right\} \\ &= \left\{ \begin{array}{c|ccccc} & d & e & f & g & \\ a & 1 & 2 & \perp & 3 & \\ b & 4 & \perp & 5 & \perp & \\ c & 6 & 7 & \perp & 8 & \end{array}, \begin{array}{c|ccc} & d & f & i \\ a & 11 & 12 & \perp \\ c & \perp & 14 & 13 \\ h & 15 & \perp & \perp \end{array} \right\}. \end{aligned}$$

In [41], the following two (non-standard) operations are defined over two EIMs.

Let the two EIMs $A = [K, L, \{a_{k_i, l_j}\}]$ and $B = [P, Q, \{b_{p_r, q_s}\}]$ be given.

The first, simpler, operation is defined by

$$A \overset{\bullet}{\underset{1}{\div}} B = [K \div P, L \div Q, \{c_{u_v, w_t}\}],$$

where here and below for every two arbitrary sets X, Y :

$$\begin{aligned} X \div Y &= (X - Y) \cup (Y - X); \\ c_{u_v, w_t} &= \begin{cases} a_{k_i, l_j}, & \text{if } u_v = k_i \in K - P \text{ and } w_t = l_j \in L - Q, \\ b_{p_r, q_s}, & \text{if } w_t = p_r \in P - K \text{ and } w_t = q_s \in Q - L, \\ \perp & \text{otherwise.} \end{cases} \end{aligned}$$

The second operation is defined by

$$A \overset{\bullet}{\underset{2}{\div}} B = [K \cup P, L \cup Q, \{c_{u_v, w_t}\}],$$

where

$$c_{u_v, w_t} = \begin{cases} a_{k_i, l_j}, & \text{if } u_v = k_i \in K - P \text{ and } w_t = l_j \in L \\ & \text{or } u_v = k_i \in K \text{ and } w_t = l_j \in L - Q, \\ b_{p_r, q_s}, & \text{if } w_t = p_r \in P - K \text{ and } w_t = q_s \in Q \\ & \text{or } w_t = p_r \in P \text{ and } w_t = q_s \in Q - L, \\ \perp & \text{otherwise.} \end{cases}$$

In [41] it is proved that:

- for every two EIMs A and B , for each operation $\circ$, and for operation $*$ defined for every two $x, y \in \mathcal{X}$ by $x * y = e_*$ there are as follows:

$$A \overset{\bullet}{\underset{1}{\div}} B = (A \ominus B) \oplus_{(\circ)} (B \ominus A),$$

$$A \overset{\bullet}{\underset{2}{\div}} B = A \oplus_{(*)} B;$$

- for every two EIMs A and B , for each operation $\circ$:

$$\begin{aligned} A \overset{\bullet}{\underset{1}{\circ}} B &= \text{pr}_{K-P, L-Q} A \oplus_{(\circ)} \text{pr}_{P-K, Q-L} B, \\ A \overset{\bullet}{\underset{2}{\circ}} B &= \text{pr}_{K-P, L} A \oplus_{(\circ)} \text{pr}_{K \cap P, L \div Q} A \oplus_{(\circ)} \text{pr}_{P-K, Q} A \\ &= \text{pr}_{K, L-Q} A \oplus_{(\circ)} \text{pr}_{K \div P, L \cap Q} A \oplus_{(\circ)} \text{pr}_{P, Q-L} A; \end{aligned}$$

- $(\mathcal{M}, \overset{\bullet}{\underset{1}{\circ}}, I_{\emptyset})$ is a commutative group;
- $(\mathcal{M}, \overset{\bullet}{\underset{2}{\circ}}, I)$ is a commutative monoid.

The visualizations of both operations are shown in Figures 2.1 and 2.2.

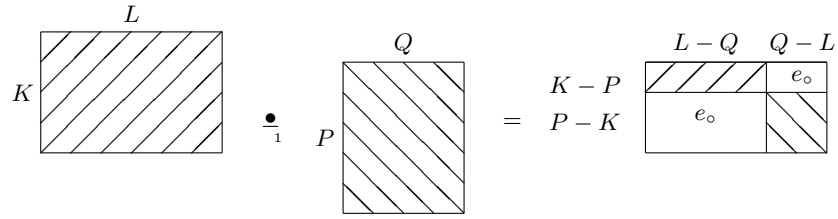


Figure 2.1. Geometrical interpretation of operation $\overset{\bullet}{\underset{1}{\circ}}$

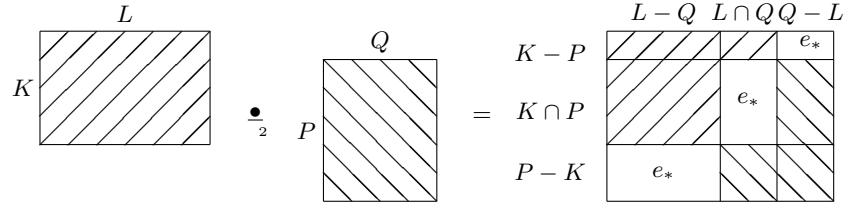


Figure 2.2. Geometrical interpretation of operation $\overset{\bullet}{\underset{2}{\circ}}$

2.4 Permanents and determinants of EIMs

As it is well-known, to each standard matrix a number can be juxtaposed, which is called determinant (see, e.g., [62, 63]). Also, it is known that if we change the places of two rows or two columns of a standard matrix, the determinant of the new matrix will coincide with the former, but with an opposite sign (i.e., sign “+” is changed with sign “−” or vice versa). The same change of two rows or two columns of an IM, however, does not change the form of the new IM. For example, the two IMs from the example above satisfy the equality

$$\begin{array}{c|ccc} & d & i & f \\ \hline a & 11 & \perp & 12 \\ c & \perp & 13 & 14 \\ h & 15 & \perp & \perp \end{array} = \begin{array}{c|ccc} & d & f & i \\ \hline a & 11 & 12 & \perp \\ c & \perp & 14 & 13 \\ h & 15 & \perp & \perp \end{array}$$

or in the case when the IM-elements are real numbers

$$\begin{array}{c|ccc} & d & i & f \\ \hline a & 11 & 0 & 12 \\ c & 0 & 13 & 14 \\ h & 15 & 0 & 0 \end{array} = \begin{array}{c|ccc} & d & f & i \\ \hline a & 11 & 12 & 0 \\ c & 0 & 14 & 13 \\ h & 15 & 0 & 0 \end{array}$$

Of course, we can juxtapose a determinant only to a matrix which elements are real or complex numbers. Having in mind the above equality, we can conclude that for IMs there is no possibility to juxtapose a determinant. On the other hand, to each IM which elements are real or complex numbers, we can juxtapose a permanent (see, e.g. [64]).

Now, we extend this possibility to each EIM. For this reason, we must define some evaluating function $\Phi : \mathcal{X} \rightarrow \mathbb{R}$, such that it is an identity in the case $\mathcal{X} = \mathbb{R}$.

Let

$$A = [K, L, \{a_{k_i, l_j}\}] = \begin{array}{c|ccccc} & l_1 & \cdots & l_j & \cdots & l_n \\ \hline k_1 & a_{k_1, l_1} & \cdots & a_{k_1, l_j} & \cdots & a_{k_1, l_n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_i & a_{k_i, l_1} & \cdots & a_{k_i, l_j} & \cdots & a_{k_i, l_n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_m & a_{k_m, l_1} & \cdots & a_{k_m, l_j} & \cdots & a_{k_m, l_n} \end{array},$$

be an EIM for the elements of which there exists an evaluating function Φ . Let for $m \leq n$ $V_n^m(l_1, \dots, l_m)$ be the set of all m -element variations of indices $l_1, \dots, l_n$ and for $m \geq n$: $V_m^n(k_1, \dots, k_m)$ be the set of all n -element variations of indices $k_1, \dots, k_n$. Then

$$\text{per}_{\Phi}(A) = \begin{cases} \sum_{[\rho_1, \dots, \rho_m] \in V_n^m(l_1, \dots, l_n)} \Phi(a_{k_1, l_{\rho_1}}) \cdots \Phi(a_{k_m, l_{\rho(m)}}), & \text{if } m = \text{card}(K) \leq \text{card}(L) = n \\ \sum_{[\sigma_1, \dots, \sigma_n] \in V_m^n(k_1, \dots, k_m)} \Phi(a_{k_{\sigma_1}, l_1}) \cdots \Phi(a_{k_{\sigma_n}, l_n}), & \text{if } m = \text{card}(K) \geq \text{card}(L) = n \end{cases}$$

or, as it is shown in [71], if $m \leq n$, then:

$$\text{per}_{\Phi}^r \langle k_1, \dots, k_m \rangle_{l_1, \dots, l_n}(A) = \sum_{[\rho_1, \dots, \rho_m] \in V_n^m(l_1, \dots, l_n)} \Phi(a_{k_1, l_{\rho_1}}) \cdots \Phi(a_{k_m, l_{\rho_m}})$$

(index “ r ” – from row), and if $m \geq n$, then

$$\text{per}_{\Phi}^c \langle k_1, \dots, k_m \rangle_{l_1, \dots, l_n}(A) = \sum_{[\sigma_1, \dots, \sigma_n] \in V_m^n(k_1, \dots, k_m)} \Phi(a_{k_{\sigma_1}, l_1}) \cdots \Phi(a_{k_{\sigma_n}, l_n})$$

(index “ c ” – from column).

And yet, we can preserve the concept of a determinant by adding an auxiliary condition. For brevity, here we discuss only the case of a square EIM, i.e., the sets K and L have equal number of elements. Let us fix the order of the elements of set K as a vector $[k_1, \dots, k_m]$. Each of its permutations $[k_{\rho_1}, \dots, k_{\rho_m}]$ will be estimated as odd or even with respect to the vector $[k_1, \dots, k_m]$. Then, the determinant of the EIM A can be defined by

$$\det_{\Phi}(A) = \begin{cases} \sum_{[\rho_1, \dots, \rho_m] \in V_n^m(l_1, \dots, l_n)} (-1)^{[\rho_1, \dots, \rho_m]} \Phi(a_{k_1, l_{\rho_1}}) \cdots \Phi(a_{k_m, l_{\rho_m}}), & \text{if } m = \text{card}(K) \leq \text{card}(L) = n; \\ \sum_{[\sigma_1, \dots, \sigma_n] \in V_m^n(k_1, \dots, k_m)} (-1)^{[\sigma_1, \dots, \sigma_n]} \Phi(a_{k_{\sigma_1}, l_1}) \cdots \Phi(a_{k_{\sigma_n}, l_n}), & \text{if } m = \text{card}(K) \geq \text{card}(L) = n, \end{cases}$$

or, as it is shown in [71], if $m \leq n$, then:

$$\det_{\Phi}^r \langle k_1, \dots, k_m \rangle_{\langle l_1, \dots, l_n \rangle} (A) = \sum_{[\rho_1, \dots, \rho_m] \in V_n^m(l_1, \dots, l_n)} (-1)^{[\rho_1, \dots, \rho_m]} \Phi(a_{k_1, l_{\rho_1}}) \cdots \Phi(a_{k_m, l_{\rho_m}})$$

and if $m \geq n$, then

$$\det_{\Phi}^c \langle k_1, \dots, k_m \rangle_{\langle l_1, \dots, l_n \rangle} (A) = \sum_{[\sigma_1, \dots, \sigma_n] \in V_m^n(k_1, \dots, k_m)} (-1)^{[\sigma_1, \dots, \sigma_n]} \Phi(a_{k_{\sigma_1}, l_1}) \cdots \Phi(a_{k_{\sigma_n}, l_n}).$$

It can be directly seen from the above definitions that for each EIM A

$$\begin{aligned} \text{per}_{\Phi}^r(A') &= \text{per}_{\Phi}^c(A), \\ \det_{\Phi}^r(A') &= \det_{\Phi}^c(A). \end{aligned}$$

In [71] some properties of the IM permanents and determinants are formulated and proved, e.g. the following Lemmas 1, 2, 3.

Lemma 1. Let all elements in one row or column of the EIM A be 0's. If $m \leq n$, then

$$\det_{\Phi}^r \langle k_1, \dots, k_m \rangle_{\langle l_1, \dots, l_n \rangle} (A) = 0 \quad \text{and} \quad \text{per}_{\Phi}^r \langle k_1, \dots, k_m \rangle_{\langle l_1, \dots, l_n \rangle} (A) = 0,$$

and if $m \geq n$, then

$$\det_{\Phi}^c \langle k_1, \dots, k_m \rangle_{\langle l_1, \dots, l_n \rangle} (A) = 0 \quad \text{and} \quad \text{per}_{\Phi}^c \langle k_1, \dots, k_m \rangle_{\langle l_1, \dots, l_n \rangle} (A) = 0.$$

Lemma 2. If λ is a real or complex number, then

$$\begin{aligned}
\lambda \text{per}_{\Phi}^r \langle k_1, \dots, k_m \rangle_{l_1, \dots, l_n} (A) &= \lambda \sum_{[\rho_1, \dots, \rho_m] \in V_n^m(l_1, \dots, l_n)} \Phi(a_{k_1, l_{\rho_1}}) \cdots \Phi(a_{k_m, l_{\rho(m)}}) \\
&= \sum_{[\rho_1, \dots, \rho_m] \in V_n^m(l_1, \dots, l_n)} \lambda \Phi(a_{k_1, l_{\rho_1}}) \cdots \Phi(a_{k_m, l_{\rho(m)}}) \\
\lambda \text{per}_{\Phi}^c \langle k_1, \dots, k_m \rangle_{l_1, \dots, l_n} (A) &= \lambda \sum_{[\sigma_1, \dots, \sigma_n] \in V_m^n(k_1, \dots, k_m)} \Phi(a_{k_{\sigma_1}, l_1}) \cdots \Phi(a_{k_{\sigma_n}, l_n}), \\
&= \sum_{[\sigma_1, \dots, \sigma_n] \in V_m^n(k_1, \dots, k_m)} \lambda \Phi(a_{k_{\sigma_1}, l_1}) \cdots \Phi(a_{k_{\sigma_n}, l_n}), \\
\lambda \det_{\Phi}^r \langle k_1, \dots, k_m \rangle_{l_1, \dots, l_n} (A) &= \lambda \sum_{[\rho_1, \dots, \rho_m] \in V_n^m(l_1, \dots, l_n)} (-1)^{[\rho_1, \dots, \rho_m]} \Phi(a_{k_1, l_{\rho_1}}) \cdots \Phi(a_{k_m, l_{\rho_m}}) \\
&= \sum_{[\rho_1, \dots, \rho_m] \in V_n^m(l_1, \dots, l_n)} (-1)^{[\rho_1, \dots, \rho_m]} \lambda \Phi(a_{k_1, l_{\rho_1}}) \cdots \Phi(a_{k_m, l_{\rho_m}}) \\
\lambda \det_{\Phi}^c \langle k_1, \dots, k_m \rangle_{l_1, \dots, l_n} (A) &= \lambda \sum_{[\sigma_1, \dots, \sigma_n] \in V_m^n(k_1, \dots, k_m)} (-1)^{[\sigma_1, \dots, \sigma_n]} \Phi(a_{k_{\sigma_1}, l_1}) \cdots \Phi(a_{k_{\sigma_n}, l_n}) \\
&= \sum_{[\sigma_1, \dots, \sigma_n] \in V_m^n(k_1, \dots, k_m)} (-1)^{[\sigma_1, \dots, \sigma_n]} \lambda \Phi(a_{k_{\sigma_1}, l_1}) \cdots \Phi(a_{k_{\sigma_n}, l_n}).
\end{aligned}$$

Lemma 3. Let λ be a real or complex number and let for every two different indices $k_x, k_y \in K$ and for all indices $l_j \in L$:

$$a_{k_x, l_j} = \lambda a_{k_y, l_j}$$

for the elements $a_{k_x, l_j}, \lambda a_{k_y, l_j}$ of the IM A . Then

$$\begin{aligned}
\text{per}_{\Phi}^r \langle k_1, \dots, k_m \rangle_{l_1, \dots, l_n} (A) &= 0, \\
\text{per}_{\Phi}^c \langle k_1, \dots, k_m \rangle_{l_1, \dots, l_n} (A) &= 0, \\
\det_{\Phi}^r \langle k_1, \dots, k_m \rangle_{l_1, \dots, l_n} (A) &= 0, \\
\det_{\Phi}^c \langle k_1, \dots, k_m \rangle_{l_1, \dots, l_n} (A) &= 0.
\end{aligned}$$

Here, we extend the results from [71] to the case of EIMs.

Corollary. The above four equalities are valid when $\lambda = 0$ (i.e., when in the EIM there is a row or column which elements are only 0's) and when $\lambda = 1$ (i.e., when in A there are two rows or columns which elements are pairwise equal).

Finally, we will mention that in [71–73] solutions of the equations

$$\begin{aligned}
A \oplus_{(\circ)} X &= B, \\
A \otimes_{(\circ)} X &= B, \\
A \ominus X &= B
\end{aligned}$$

are obtained, when A, B and X are IMs over the set of real numbers and when they are IFIMs. At the moment, an **Open Problem** is: what are the solutions of the equality

$$A \odot_{(\circ,*)} X = B$$

when A, B and X are from one of both above types: IMs whose elements are real numbers or IFIMs.

2.5 Relations over EIMs

Let the two IMs $A = [K, L, \{a_{k,l}\}]$ and $B = [P, Q, \{b_{p,q}\}]$ be given, where $a_{k,l} \in \mathcal{X}$, $b_{p,q} \in \mathcal{Y}$, and $K, L, P, Q \subset \mathcal{I}$. Let R_s and R_n be a strict and a non-strict relations over $\mathcal{X} \times \mathcal{Y}$, respectively.

We introduce the following definitions, where $\subset$ and $\subseteq$ denote the relations “strong inclusion” and “weak inclusion” over standard sets.

The first two relations from Section 1.3 keep their form, while, the rest four definitions are valid only if there are two relations R_s and R_n – a strict and a non-strict one – defined over $\mathcal{X} \times \mathcal{Y}$.

Then, for EIMs, the definitions for relations over EIMs obtain respectively one of the following forms.

The strict relation “inclusion about value” is

$$A \subset_v B \text{ iff } (K = P) \& (L = Q) \& (\forall k \in K)(\forall l \in L)(R_s(a_{k,l}, b_{k,l})).$$

The non-strict relation “inclusion about value” is

$$A \subseteq_v B \text{ iff } (K = P) \& (L = Q) \& (\forall k \in K)(\forall l \in L)(R_n(a_{k,l}, b_{k,l})).$$

The strict relation “inclusion” is

$$\begin{aligned} A \subset_* B \text{ iff } & ((K \subset P) \& (L \subset Q)) \vee ((K \subseteq P) \& (L \subset Q)) \\ & \vee ((K \subset P) \& (L \subseteq Q)) \& (\forall k \in K)(\forall l \in L)(R_s(a_{k,l}, b_{k,l})). \end{aligned}$$

The non-strict relation “inclusion” is

$$A \subseteq_* B \text{ iff } (K \subseteq P) \& (L \subseteq Q) \& (\forall k \in K)(\forall l \in L)(R_n(a_{k,l}, b_{k,l})).$$

2.6 Hierarchical IMs

A special case of EIMs are IMs some (or all) elements of which are also IMs. Following [14], we will call these IMs *hierarchical IMs (HIMs)*. Obviously, none of the different types IMs listed in Chapter 1 can be a HIM.

In [20], two **hierarchical operators** are defined over HIMs. They are applicable onto HIMs, i.e., onto EIM, when their elements are not only numbers, variables, etc., but also whole IMs. First, we give the definitions of these

operators and after this we will define a new form of a hierarchical operator over HIMs. In Chapter 3 we will repeat the definition of this operator over an n -dimensional HIM.

Let A be a HIM and let its element a_{k_f, l_g} be an IM by itself:

$$a_{k_f, l_g} = [P, Q, \{b_{p_r, q_s}\}],$$

where $1 \leq r \leq x, 1 \leq s \leq y$ and

$$K \cap P = L \cap Q = \emptyset.$$

The first hierarchical operator is

$$A|(a_{k_f, l_g}) = [(K - \{k_f\}) \cup P, (L - \{l_g\}) \cup Q, \{c_{t_u, v_w}\}],$$

where

$$c_{t_u, v_w} = \begin{cases} a_{k_i, l_j}, & \text{if } t_u = k_i \in K - \{k_f\} \text{ and } v_w = l_j \in L - \{l_g\}, \\ b_{p_r, q_s}, & \text{if } t_u = p_r \in P \text{ and } v_w = q_s \in Q, \\ \perp, & \text{otherwise} \end{cases},$$

and when a_{k_i, l_j} are real numbers, the symbol $\perp$ can be changed with 0.

Let us assume that in the case when a_{k_f, l_g} is not an element of IM A , then

$$A|(a_{k_f, l_g}) = A.$$

From the first definition of a hierarchical operator it follows that

$$A|(a_{k_f, l_g}) = \begin{array}{c|cccccccc} & l_1 & \cdots & l_{g-1} & q_1 & \cdots & q_y & l_{g+1} & \cdots & l_n \\ \hline k_1 & a_{k_1, l_1} & \cdots & a_{k_1, l_{g-1}} & 0 & \cdots & 0 & a_{k_1, l_{g+1}} & \cdots & a_{k_1, l_n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ k_{f-1} & a_{k_{f-1}, l_1} & \cdots & a_{k_{f-1}, l_{g-1}} & 0 & \cdots & 0 & a_{k_{f-1}, l_{g+1}} & \cdots & a_{k_{f-1}, l_n} \\ p_1 & 0 & \cdots & 0 & b_{p_1, q_1} & \cdots & b_{p_1, q_y} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ p_x & 0 & \cdots & 0 & b_{p_x, q_1} & \cdots & b_{p_x, q_y} & 0 & \cdots & 0 \\ k_{f+1} & a_{k_{f+1}, l_1} & \cdots & a_{k_{f+1}, l_{g-1}} & 0 & \cdots & 0 & a_{k_{f+1}, l_{g+1}} & \cdots & a_{k_{f+1}, l_n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ k_m & a_{k_m, l_1} & \cdots & a_{k_m, l_{g-1}} & 0 & \cdots & 0 & a_{k_m, l_{g+1}} & \cdots & a_{k_m, l_n} \end{array}.$$

From this form of the IM $A|(a_{k_f, l_g})$ we see that for the hierarchical operator the following assertion holds. Let A be a HIM and let a_{k_f, l_g} be its element. Then

$$A|(a_{k_f, l_g}) = (A \ominus [\{k_f\}, \{l_g\}, \{0\}]) \oplus a_{k_f, l_g}.$$

In [20], the first hierarchical operator is modified so that all the information from the HIMs, participating in it, is preserved. The new – second – form of this operator for the above defined HIM A and its fixed element is:

$$A|^*(a_{k_f, l_g})$$

	l_1	$\cdots$	l_{g-1}	q_1	$\cdots$	q_y	l_{g+1}	$\cdots$	l_n
k_1	a_{k_1, l_1}	$\cdots$	$a_{k_1, l_{g-1}}$	a_{k_1, l_g}	$\cdots$	a_{k_1, l_g}	$a_{k_1, l_{g+1}}$	$\cdots$	a_{k_1, l_n}
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
k_{f-1}	a_{k_{f-1}, l_1}	$\cdots$	$a_{k_{f-1}, l_{g-1}}$	a_{k_{f-1}, l_g}	$\cdots$	a_{k_{f-1}, l_g}	$a_{k_{f-1}, l_{g+1}}$	$\cdots$	a_{k_{f-1}, l_n}
p_1	a_{k_f, l_1}	$\cdots$	$a_{k_f, l_{g-1}}$	b_{p_1, q_1}	$\cdots$	b_{p_1, q_y}	$a_{k_f, l_{g+1}}$	$\cdots$	a_{k_f, l_n}
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
p_x	a_{k_f, l_1}	$\cdots$	$a_{k_f, l_{g-1}}$	b_{p_x, q_1}	$\cdots$	b_{p_x, q_y}	$a_{k_f, l_{g+1}}$	$\cdots$	a_{k_f, l_n}
k_{f+1}	a_{k_{f+1}, l_1}	$\cdots$	$a_{k_{f+1}, l_{g-1}}$	a_{k_{f+1}, l_g}	$\cdots$	a_{k_{f+1}, l_g}	$a_{k_{f+1}, l_{g+1}}$	$\cdots$	a_{k_{f+1}, l_n}
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
k_m	a_{k_m, l_1}	$\cdots$	$a_{k_m, l_{g-1}}$	a_{k_m, l_g}	$\cdots$	a_{k_m, l_g}	$a_{k_m, l_{g+1}}$	$\cdots$	a_{k_m, l_n}

Now, the following assertion is valid. Let A be a HIM and let a_{k_f, l_g} be its element. Then

$$A|^*(a_{k_f, l_g}) = (A \ominus [\{k_f\}, \{l_g\}, \{0\}]) \oplus a_{k_f, l_g} \oplus [P, L - \{l_g\}, \{c_{z', l_j}\}] \oplus [K - \{k_f\}, Q, \{d_{k_i, z''}\}],$$

where for each $z' \in P$ and for each $l_j \in L - \{l_g\}$,

$$c_{z', l_j} = a_{k_f, l_j}$$

and for each $k_i \in K - \{k_f\}$ and for each $z'' \in Q$,

$$d_{k_i, z''} = a_{k_i, l_g}.$$

We, following [20], can give other representations of IMs $A|(a_{k_f, l_g})$ and $A|^*(a_{k_f, l_g})$, using other operations defined over IMs.

Let A be an IM and let a_{k_f, l_g} be its element. Then we can prove that the following equalities are valid:

$$A|(a_{k_f, l_g}) = \text{pr}_{K - \{k_f\}, L - \{l_g\}} A \oplus a_{k_f, l_g},$$

$$A|^*(a_{k_f, l_g}) = \text{pr}_{K - \{k_f\}, L - \{l_g\}} A \oplus a_{k_f, l_g} \oplus [P, L - \{l_g\}, \{c_{x, l_j}\}] \oplus [K - \{k_f\}, Q, \{d_{k_i, y}\}],$$

where c_{z', l_j} and $d_{k_i, z''}$ satisfy the above equalities.

Here, following [28], we discuss a 2-dimensional HIM each element of which is a HIM and an operator that modifies this HIM to a standard EIM.

Let $A = [K, L, \{a_{k_i, l_j}\}]$ be an HIM, where $K = \{k_1, k_2, \dots, k_m\}$, $L = \{l_1, l_2, \dots, l_n\}$, and for $1 \leq i \leq m$, and $1 \leq j \leq n : a_{k_i, l_j} \in \mathcal{X}$. Let each its element a_{k_f, l_g} be a HIM by itself:

$$a_{k_f, l_g} = [P_{k_f, l_g}, Q_{k_f, l_g}, \{b_{k_f, l_g, p_u, q_v}\}],$$

where $1 \leq f \leq m$, $1 \leq g \leq n$, $1 \leq u \leq r_{f, g}$, $1 \leq v \leq s_{f, g}$ and

$$P_{k_f, l_g} = \{p_{k_f, l_g, 1}, \dots, p_{k_f, l_g, r_{f, g}}\},$$

$$Q_{k_f, l_g} = \{q_{k_f, l_g, 1}, \dots, q_{k_f, l_g, s_{f, g}}\},$$

$$K \cap P_{k_f, l_g} = L \cap Q_{k_f, l_g} = \emptyset \quad (2.1)$$

and for every four indices $k_f, k_h, l_g, l_i \in \mathcal{I}$:

$$P_{k_f, l_g} \cap P_{k_h, l_i} = Q_{k_f, l_g} \cap Q_{k_h, l_i} = \emptyset. \quad (2.2)$$

The new (third) hierarchical operator is defined by

$$A|^* =$$

	$q_{k_1, l_1, 1}$	$\dots$	$q_{k_1, l_1, s_{1,1}}$	$\dots$	$q_{k_m, l_n, 1}$	$\dots$	$q_{k_m, l_n, s_{m,n}}$
$p_{k_1, l_1, 1}$	$a_{k_1, l_1, 1, 1}$	$\dots$	$a_{k_1, l_1, 1, s_{1,1}}$	$\dots$	0	$\dots$	0
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$
$p_{k_1, l_1, r_{1,1}}$	$a_{k_1, l_1, r_{1,1}, 1}$	$\dots$	$a_{k_1, l_1, r_{1,1}, s_{1,1}}$	$\dots$	0	$\dots$	0
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$
$p_{k_m, l_n, 1}$	0	$\dots$	0	$\dots$	$a_{k_m, l_n, 1, 1}$	$\dots$	$a_{k_m, l_n, 1, s_{m,n}}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$p_{k_m, l_n, r_{m,n}}$	0	$\dots$	0	$\dots$	$a_{k_m, l_n, r_{m,n}, 1}$	$\dots$	$a_{k_m, l_n, r_{m,n}, s_{m,n}}$

We see that conditions (2.1) and (2.2) are important for the correctness of the definition.

For example, if

$$A = \begin{array}{c|cc} & l_1 & l_2 \\ \hline k_1 & a_{1,1} & a_{1,2} \\ k_2 & a_{2,1} & a_{2,2} \\ k_3 & a_{3,1} & a_{3,2} \end{array},$$

where

$$a_{1,1} = \begin{array}{c|cc} & q_{1,1} & q_{1,2} \\ \hline p_{1,1} & b_{1,1} & b_{1,2} \\ p_{1,2} & b_{2,1} & b_{2,2} \\ p_{1,3} & b_{3,1} & b_{3,2} \end{array}, \quad a_{1,2} = \begin{array}{c|ccc} & q_{2,1} & q_{2,2} & q_{2,3} \\ \hline p_{2,1} & c_{1,1} & c_{1,2} & c_{1,3} \\ p_{2,2} & c_{2,1} & c_{2,2} & c_{2,3} \end{array},$$

$$a_{2,1} = \begin{array}{c|cc} & q_{3,1} & \\ \hline p_{3,1} & d_{1,1} & \end{array}, \quad a_{2,2} = \begin{array}{c|ccc} & q_{4,1} & q_{4,2} & \\ \hline p_{4,1} & e_{1,1} & e_{1,2} & \\ p_{4,2} & e_{2,1} & e_{2,2} & \\ p_{4,3} & e_{3,1} & e_{3,2} & \end{array},$$

$$a_{3,1} = \frac{\overline{p_{5,1}}}{p_{5,2}} \left| \begin{array}{c} q_{5,1} \\ f_{1,1} \\ f_{2,1} \end{array} \right|, \quad a_{3,2} = \frac{\overline{p_{6,1}}}{p_{6,1}} \left| \begin{array}{cc} q_{6,1} & q_{6,2} \\ g_{1,1} & g_{1,2} \end{array} \right|,$$

then

$$A|^* = \begin{array}{c|cccccccccccc} & q_{1,1} & q_{1,2} & q_{2,1} & q_{2,2} & q_{2,3} & q_{3,1} & q_{4,1} & q_{4,2} & q_{5,1} & q_{6,1} & q_{6,2} \\ \hline p_{1,1} & b_{1,1} & b_{1,2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ p_{1,2} & b_{2,1} & b_{2,2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ p_{1,3} & b_{3,1} & b_{3,2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ p_{2,1} & 0 & 0 & c_{1,1} & c_{1,2} & c_{1,3} & 0 & 0 & 0 & 0 & 0 & 0 \\ p_{2,2} & 0 & 0 & c_{2,1} & c_{2,2} & c_{2,3} & 0 & 0 & 0 & 0 & 0 & 0 \\ p_{3,1} & 0 & 0 & 0 & 0 & 0 & d_{1,1} & 0 & 0 & 0 & 0 & 0 \\ p_{4,1} & 0 & 0 & 0 & 0 & 0 & 0 & e_{1,1} & e_{1,2} & 0 & 0 & 0 \\ p_{4,2} & 0 & 0 & 0 & 0 & 0 & 0 & e_{2,1} & e_{2,2} & 0 & 0 & 0 \\ p_{4,3} & 0 & 0 & 0 & 0 & 0 & 0 & e_{3,1} & e_{3,2} & 0 & 0 & 0 \\ p_{5,1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & f_{1,1} & 0 & 0 \\ p_{5,2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & f_{2,1} & 0 & 0 \\ p_{6,1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g_{1,1} & g_{1,2} \end{array}.$$

Now, keeping condition (2.1), we can change condition (2.2) with the weaker one: for every four indices $k_f, k_h \in \mathcal{I}^*$ and $l_g, l_i \in \mathcal{I}^*$, and $\times$ the standard Cartesian product,

$$P_{k_f, l_g} \times Q_{k_f, l_g} \cap P_{k_h, l_i} \times Q_{k_h, l_i} = \emptyset,$$

i.e., there is no pair of indices that is found in two different a -IMs that are elements of the given HIM A .

In this case, if there are two or more p -indices in $A|^*$ that coincide, all members of the rows with these indices are written on the respective places in the first row with coinciding indices. Obviously, there are no two elements, that stay on one and the same place, because they are members of different HIMs.

After this, if there are two or more q -indices in $A|^*$ that coincide, all members of the columns with these indices are written on the respective places in the first column with coinciding indices. Obviously, again, there are no two elements, that stay on one and the same place.

For example, for the HIM A from the above example, if

$$a_{1,2} = \frac{\overline{p_{1,2}}}{p_{2,2}} \left| \begin{array}{ccc} q_{2,1} & q_{2,2} & q_{2,3} \\ c_{1,1} & c_{1,2} & c_{1,3} \\ c_{2,1} & c_{2,2} & c_{2,3} \end{array} \right|, \quad a_{2,2} = \frac{\overline{p_{4,1}}}{p_{4,3}} \left| \begin{array}{cc} q_{3,1} & q_{4,2} \\ e_{1,1} & e_{1,2} \\ e_{2,1} & e_{2,2} \\ e_{3,1} & e_{3,2} \end{array} \right|,$$

$$a_{3,1} = \frac{\overline{p_{3,1}}}{p_{3,2}} \left| \begin{array}{c} q_{2,1} \\ f_{1,1} \\ f_{2,1} \end{array} \right|, \quad a_{3,2} = \frac{\overline{p_{6,1}}}{p_{6,1}} \left| \begin{array}{cc} q_{1,1} & q_{1,2} \\ g_{1,1} & g_{1,2} \end{array} \right|,$$

then

$$A|^* = \begin{array}{c|cccccccc} & q_{1,1} & q_{1,2} & q_{2,1} & q_{2,2} & q_{2,3} & q_{3,1} & q_{4,1} & q_{4,2} \\ \hline p_{1,1} & b_{1,1} & b_{1,2} & 0 & 0 & 0 & 0 & 0 & 0 \\ p_{1,2} & b_{2,1} & b_{2,2} & 0 & 0 & 0 & 0 & 0 & 0 \\ p_{1,3} & b_{3,1} & b_{3,2} & 0 & 0 & 0 & 0 & 0 & 0 \\ p_{2,1} & 0 & 0 & c_{1,1} & c_{1,2} & c_{1,3} & 0 & 0 & 0 \\ p_{2,2} & 0 & 0 & c_{2,1} & c_{2,2} & c_{2,3} & 0 & 0 & 0 \\ p_{3,1} & 0 & 0 & 0 & f_{1,1} & 0 & d_{1,1} & 0 & 0 \\ p_{3,2} & 0 & 0 & 0 & f_{2,1} & 0 & 0 & e_{1,1} & e_{1,2} \\ p_{4,2} & 0 & 0 & 0 & 0 & 0 & 0 & e_{2,1} & e_{2,2} \\ p_{4,3} & 0 & 0 & 0 & 0 & 0 & 0 & e_{3,1} & e_{3,2} \\ p_{6,1} & g_{1,1} & g_{1,2} & 0 & 0 & 0 & 0 & 0 & 0 \end{array}.$$

The next two assertions are proved in [28].

Let $A = [K, L, \{a_{k_i, l_j}\}]$ be an HIM and let each of its elements a_{k_f, l_g} be an IM. Then

$$A|^* = \sum_{\substack{1 \leq f \leq m \\ 1 \leq g \leq n}} a_{k_f, l_g},$$

where symbol $\sum$ denotes the generalization of the operation addition $\oplus_{(\circ)}$. Here, the sub-operation $\circ$ of the operation addition $\oplus_{(\circ)}$ is omitted from $\sum$, because it will be applied over no pair of elements of the a -IMs.

2.7 Examples from graph theory. Part 1

Let us have the following oriented graph in Figure 2.3.

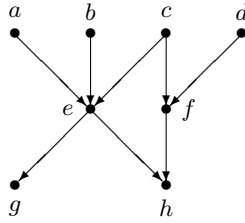


Figure 2.3. An example of an oriented graph

For it, we can construct the adjacency matrix of the graph as an IM whose elements are from the set $\{0, 1\}$:

$$A = \begin{array}{c|cccccccc} & a & b & c & d & e & f & g & h \\ \hline a & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ b & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ c & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ d & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ e & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ f & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ g & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ h & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} .$$

Obviously, the IM A in this form is more complex than the standard (adjacency) matrix representing this graph, because both matrices are coded by 64 symbols with 0's or 1's, but the IM has additional 16 alphabetic symbols – the indices of its rows and columns. But having in mind that the columns indexed by a, b, c, d and the rows, indexed by g, h contain only zeros and do not give any important information, we can transform the adjacency IM to the form

$$B = \begin{array}{c|cccc} & e & f & g & h \\ \hline a & 1 & 0 & 0 & 0 \\ b & 1 & 0 & 0 & 0 \\ c & 1 & 0 & 0 & 0 \\ d & 0 & 1 & 1 & 0 \\ e & 0 & 0 & 1 & 1 \\ f & 0 & 0 & 0 & 1 \end{array} ,$$

in which we omit the columns for the initial vertices and the rows for the final vertices, respectively. This new $(0, 1)$ -IM B can be called “reduced adjacency IM”. Obviously, it contains only 34 symbols (24 symbols with 0's or 1's and 10 alphabet symbols). Therefore, the use of IMs for graph representation of is appropriate. In [20] it is shown that the IM representation of the graphs preserves all possibilities of standard matrix representation. So, below we will use IMs.

Obviously, the demonstrated reduction is impossible for non-oriented graphs, but if we have a graph with two or more components, then the IM-representation will again be better. For example, if we have the two-component graph from Figure 2.4, it will again have a standard 8×8 adjacency matrix, while its representation of an adjacency IM will be

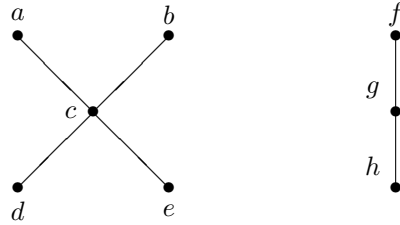


Figure 2.4. An example of a two-component graph

$$\begin{array}{c|ccccc} & a & b & c & d & e \\ \hline a & 0 & 0 & 1 & 0 & 0 \\ b & 0 & 0 & 1 & 0 & 0 \\ c & 1 & 1 & 0 & 1 & 1 \\ d & 0 & 0 & 1 & 0 & 0 \\ e & 0 & 0 & 1 & 0 & 0 \end{array} \oplus_{(\max)} \begin{array}{c|ccc} & f & g & h \\ \hline f & 0 & 1 & 0 \\ g & 1 & 0 & 1 \\ h & 0 & 1 & 0 \end{array},$$

i.e., in the first case the matrix will again contains 64 elements (digits 0 or 1), while in the second case – only 50 elements ($34 = 25 + 9$ digits and $16 = 10 + 6$ letters).

An important question is whether the IM-representation of the graphs is correct. Really, we see how the immediate neighbouring vertices of the graph are connected, but we must check the basic property of the standard adjacency matrix X , that the elements of the multiplication $X^2 = X \times X$ represent the connections between the vertices through one step (see, e.g., [60]). Using the operation $\odot_{(\max, \min)}$ we obtain for the above $(0, 1)$ -IM B

$$B^2 = B \odot_{(\max, \min)} B$$

$$\begin{array}{c|cccc} & e & f & g & h \\ \hline a & 1 & 0 & 0 & 0 \\ b & 1 & 0 & 0 & 0 \\ c & 1 & 0 & 0 & 0 \\ d & 0 & 1 & 1 & 0 \\ e & 0 & 0 & 1 & 1 \\ f & 0 & 0 & 0 & 1 \end{array} \odot_{(\max, \min)} \begin{array}{c|cccc} & e & f & g & h \\ \hline a & 1 & 0 & 0 & 0 \\ b & 1 & 0 & 0 & 0 \\ c & 1 & 0 & 0 & 0 \\ d & 0 & 1 & 1 & 0 \\ e & 0 & 0 & 1 & 1 \\ f & 0 & 0 & 0 & 1 \end{array} = \begin{array}{c|cccc} & e & f & g & h \\ \hline a & 0 & 0 & 1 & 1 \\ b & 0 & 0 & 1 & 1 \\ c & 0 & 0 & 1 & 1 \\ d & 0 & 0 & 0 & 1 \\ e & 0 & 0 & 0 & 0 \\ f & 0 & 0 & 0 & 0 \end{array}.$$

Obviously, this result corresponds to the result for the standard adjacency matrix. Now, as above, we can decrease the latest IM to the form

$$B^2 = \begin{array}{c|cc} & g & h \\ \hline a & 1 & 1 \\ b & 1 & 1 \\ c & 1 & 1 \\ d & 0 & 1 \end{array}$$

but in this reduction we lose information about the vertices e and f . So, it will be more appropriate to use a weaker reduction in the form

$$B^2 = \begin{array}{c|cc} & g & h \\ \hline a & 1 & 1 \\ b & 1 & 1 \\ c & 1 & 1 \\ d & 0 & 1 \\ e & 0 & 0 \\ f & 0 & 0 \end{array}$$

or

$$B^2 = \begin{array}{c|cccc} & e & f & g & h \\ \hline a & 0 & 0 & 1 & 1 \\ b & 0 & 0 & 1 & 1 \\ c & 0 & 0 & 1 & 1 \\ d & 0 & 0 & 0 & 1 \end{array}.$$

Obviously, in both cases we use equal (8) number of letters, but in the first case we use 12 digits, while in the second case – 16.

Now, we illustrate the results of the applications of different operations over $(0,1)$ -IMs, that represent some oriented graphs. Let us have the graph C from Figure 2.5.

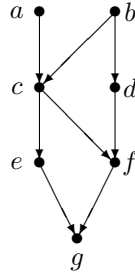


Figure 2.5. Graph C

Its $(0,1)$ -IM (in the reduced form, i.e., after omission of the row indexed by g) is

$$C = \begin{array}{c|ccccc} & c & d & e & f & g \\ \hline a & 1 & 0 & 0 & 0 & 0 \\ b & 1 & 1 & 0 & 0 & 0 \\ c & 0 & 0 & 1 & 1 & 0 \\ d & 0 & 0 & 0 & 1 & 0 \\ e & 0 & 0 & 0 & 0 & 1 \\ f & 0 & 0 & 0 & 0 & 1 \end{array}.$$

Then, we calculate:

$$C^2 = C \odot_{(\max, \min)} C = \begin{array}{c|ccccc} & c & d & e & f & g \\ \hline a & 0 & 0 & 1 & 1 & 0 \\ b & 0 & 0 & 1 & 1 & 0 \\ c & 0 & 0 & 0 & 0 & 1 \\ d & 0 & 0 & 0 & 0 & 1 \\ e & 0 & 0 & 0 & 0 & 0 \\ f & 0 & 0 & 0 & 0 & 0 \end{array}.$$

As in the previous example, we can reduce the $(0,1)$ -IM C^2 to the form

$$C^2 = \begin{array}{c|ccc} & e & f & g \\ \hline a & 1 & 1 & 0 \\ b & 1 & 1 & 0 \\ c & 0 & 0 & 1 \\ d & 0 & 0 & 1 \end{array}.$$

It is interesting to note that

$$C \odot_{(\max, \min)} C^2 = \begin{array}{c|ccccc} & c & d & e & f & g \\ \hline a & 0 & 0 & 0 & 0 & 1 \\ b & 0 & 0 & 0 & 0 & 1 \\ c & 0 & 0 & 0 & 0 & 0 \\ d & 0 & 0 & 0 & 0 & 0 \\ e & 0 & 0 & 0 & 0 & 0 \\ f & 0 & 0 & 0 & 0 & 0 \end{array} = C^2 \odot_{(\max, \min)} C.$$

Therefore,

$$C \odot_{(\max, \min)} C^2 = C^3 = C^2 \odot_{(\max, \min)} C.$$

We see that the resultant (0,1)-IM can be reduced to

$$C^3 = \begin{array}{c|c} & g \\ \hline a & 1 \\ b & 1 \end{array}.$$

Of course, in this case we lose information about the remaining vertices of the graph.

Let us have the oriented graph D from Figure 2.6.

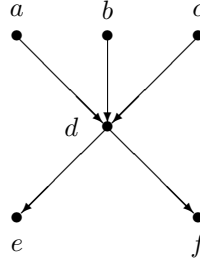


Figure 2.6. Graph D

Its adjacency (0,1)-IM is

$$D = \begin{array}{c|cccccc} & a & b & c & d & e & f \\ \hline a & 0 & 0 & 0 & 1 & 0 & 0 \\ b & 0 & 0 & 0 & 1 & 0 & 0 \\ c & 0 & 0 & 0 & 1 & 0 & 0 \\ d & 0 & 0 & 0 & 0 & 1 & 1 \\ e & 0 & 0 & 0 & 0 & 0 & 0 \\ f & 0 & 0 & 0 & 0 & 0 & 0 \end{array}.$$

Let us construct a graph, that is a result of operation substitution $\left[\begin{smallmatrix} h & i \\ c & f \end{smallmatrix}; \begin{smallmatrix} h & i \\ c & f \end{smallmatrix} \right]$ over graph D , then we obtain the graph E from Figure 2.7 with adjacency $(0, 1)$ -IM:

$$E = \begin{array}{c|cccccc} & a & b & h & d & e & i \\ \hline a & 0 & 0 & 0 & 1 & 0 & 0 \\ b & 0 & 0 & 0 & 1 & 0 & 0 \\ h & 0 & 0 & 0 & 1 & 0 & 0 \\ d & 0 & 0 & 0 & 0 & 1 & 1 \\ e & 0 & 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 & 0 & 0 \end{array} = \begin{array}{c|ccc} & d & e & i \\ \hline a & 1 & 0 & 0 \\ b & 1 & 0 & 0 \\ h & 1 & 0 & 0 \\ d & 0 & 1 & 1 \end{array}$$

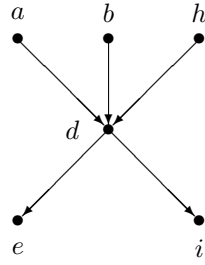


Figure 2.7. Graph E

If we would like to unite the graphs C and E , we obtain the graph with the form from Figure 2.8 and with the (reduced) $(0, 1)$ -IM

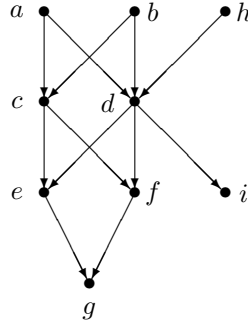


Figure 2.8. Graph $C \oplus_{(\max)} E$

$$C \oplus_{(\max)} E = \begin{array}{c|cccccc} & c & d & e & f & g & i \\ \hline a & 1 & 1 & 0 & 0 & 0 & 0 \\ b & 1 & 1 & 0 & 0 & 0 & 0 \\ c & 0 & 0 & 1 & 1 & 0 & 0 \\ d & 0 & 0 & 1 & 1 & 0 & 1 \\ e & 0 & 0 & 0 & 0 & 1 & 0 \\ f & 0 & 0 & 0 & 0 & 1 & 0 \\ h & 0 & 1 & 0 & 0 & 0 & 0 \end{array}.$$

The $(0, 1)$ -IM that is a result of operation “termwise multiplication” with sub-operations “max” and “min”, respectively, over $(0, 1)$ -IMs C and E is

$$C \otimes_{(\max)} E = \begin{array}{c|cc} & d & e \\ \hline a & 1 & 0 \\ b & 1 & 0 \\ d & 0 & 1 \end{array}, \quad \text{and} \quad C \otimes_{(\min)} E = \begin{array}{c|cc} & d & e \\ \hline a & 0 & 0 \\ b & 1 & 0 \\ d & 0 & 1 \end{array}$$

and these IMs have, respectively, the graph-forms from Figure 2.9.

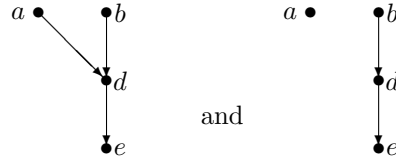


Figure 2.9. Graph $C \otimes_{(\max)} E$ and graph $C \otimes_{(\min)} E$

In addition, we mention that the graph-representation of the $(0, 1)$ -IM

$$C \oplus_{(\min)} E = \begin{array}{c|cccccc} & c & d & e & f & g & i \\ \hline a & 0 & 0 & 0 & 0 & 0 & 0 \\ b & 0 & 1 & 0 & 0 & 0 & 0 \\ c & 0 & 0 & 0 & 0 & 0 & 0 \\ d & 0 & 0 & 1 & 0 & 0 & 0 \\ e & 0 & 0 & 0 & 0 & 0 & 0 \\ f & 0 & 0 & 0 & 0 & 0 & 0 \\ h & 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$

and in the reduced form

$$C \oplus_{(\min)} E = \begin{array}{c|cc} & d & e \\ \hline b & 1 & 0 \\ d & 0 & 1 \end{array}$$

i.e., the graph form is given in Figure 2.10.


 Figure 2.10. Graph $C \oplus_{(\min)} E$

The operation $\ominus$ can be illustrated, e.g., as follows:

$$C \ominus E = \begin{array}{c|ccc} & c & f & g \\ \hline c & 0 & 1 & 0 \\ e & 0 & 0 & 1 \\ f & 0 & 0 & 1 \end{array}$$

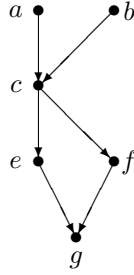
while

$$E \ominus C = \begin{array}{c|c} & i \\ \hline h & 0 \end{array}.$$

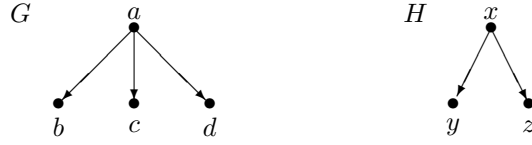
If we would like to omit some vertex of a graph, we can do this, using operation “reduction”. For example,

$$C_{(d,d)} = \begin{array}{c|cccc} & c & e & f & g \\ \hline a & 1 & 0 & 0 & 0 \\ b & 1 & 0 & 0 & 0 \\ c & 0 & 1 & 1 & 0 \\ e & 0 & 0 & 0 & 1 \\ f & 0 & 0 & 0 & 1 \end{array}.$$

This $(0, 1)$ -IM has the graph-representation from Figure 2.11.

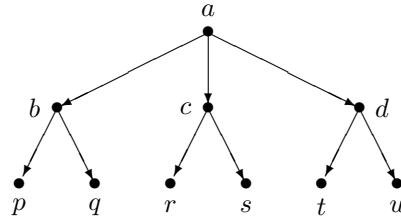

 Figure 2.11. Graph $C_{(d,d)}$

Now, we illustrate operation $\odot_{(\circ,*)}$. Let us have graphs G and H from Figure 2.12.

Figure 2.12. Graphs G and H

Let us add to each of the lower vertices of G new graphs with the form of H and let the vertices of these new graphs be the triples (b, p, q) , (c, r, s) and (d, t, u) that will replace vertices (x, y, z) , respectively. Then, we obtain the graph J from Figure 2.13, which has the $(0, 1)$ -IM-representation

$$J = G \odot_{(\circ, *)} \left(\left[\begin{array}{ccc} b & p & q \\ x & y & z \end{array} \right] H \oplus_{(\max)} \left[\begin{array}{ccc} c & r & s \\ x & y & z \end{array} \right] H \oplus_{(\max)} \left[\begin{array}{ccc} d & t & u \\ x & y & z \end{array} \right] H \right).$$

Figure 2.13. Graph J

Of course, if, for example, the graph C is not oriented, then its adjacency $(0, 1)$ -IM has the form

	a	b	c	d	e	f
a	0	0	0	1	0	0
b	0	0	0	1	0	0
c	0	0	0	1	0	0
d	1	1	1	0	1	1
e	0	0	0	1	0	0
f	0	0	0	1	0	0

while, if we add to it, for example, the arc (a, e) , its adjacency $(0, 1)$ -IM obtains the form

	a	b	c	d	e	f
a	0	0	0	1	1	0
b	0	0	0	1	0	0
c	0	0	0	1	0	0
d	1	1	1	0	1	1
e	1	0	0	1	0	0
f	0	0	0	1	0	0

If the graph has a loop, e.g., (b, b) , then its adjacency $(0, 1)$ -IM has the form

	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>	<i>e</i>	<i>f</i>
<i>a</i>	0	0	0	1	1	0
<i>b</i>	0	1	0	1	0	0
<i>c</i>	0	0	0	1	0	0
<i>d</i>	1	1	1	0	1	1
<i>e</i>	1	0	0	1	0	0
<i>f</i>	0	0	0	1	0	0

Finally, let us have a graph, for example, with only three vertices a, b, c and three arcs and let the arcs be coloured *blue*, *green* and *red*. Then its representation will be

$$G = \begin{array}{c|ccc} & a & b & c \\ \hline a & \perp & blue & red \\ b & blue & \perp & green \\ c & red & green & \perp \end{array}.$$

It is obvious that IM G is not numerical, i.e., it does not have the type of matrices considered in Chapter 1, and is only like the IMs considered in this chapter, i.e., it is an EIM.

n -Dimensional Index Matrices (n -DIM): definitions and examples

Following [20], we will first give a definition of a 3-dimensional index matrix and after this we will extend it to n dimensions for $n \geq 4$. This part of the present chapter contains new results or ones published after [20]. All IMs here can be considered as EIMs.

3.1 3-DIMs

Let $\mathcal{I}$ be a fixed set of indices and $\mathcal{X}$ be a fixed set of objects. Following [19,20], we call “3-DIM” with index sets K, L and H ($K, L, H \subset \mathcal{I}$) the object:

$$[K, L, H, \{a_{k_i, l_j, h_g}\}] \equiv \left\{ \begin{array}{c|cccc} h_g & l_1 & \cdots & l_j & \cdots & l_n \\ \hline k_1 & a_{k_1, l_1, h_g} & \vdots & a_{k_1, l_j, h_g} & \cdots & a_{k_1, l_n, h_g} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_m & a_{k_m, l_1, h_g} & \cdots & a_{k_m, l_j, h_g} & \cdots & a_{k_m, l_n, h_g} \end{array} \mid h_g \in H \right\}$$

$$\equiv \left\{ \begin{array}{c|cccc} h_1 & l_1 & \cdots & l_j & \cdots & l_n \\ \hline k_1 & a_{k_1, l_1, h_1} & \vdots & a_{k_1, l_j, h_1} & \cdots & a_{k_1, l_n, h_1} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_m & a_{k_m, l_1, h_1} & \cdots & a_{k_m, l_j, h_1} & \cdots & a_{k_m, l_n, h_1} \end{array} \right\},$$

$$\begin{array}{c|cccc} h_2 & l_1 & \cdots & l_j & \cdots & l_n \\ \hline k_1 & a_{k_1, l_1, h_2} & \vdots & a_{k_1, l_j, h_2} & \cdots & a_{k_1, l_n, h_2} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_m & a_{k_m, l_1, h_2} & \cdots & a_{k_m, l_j, h_2} & \cdots & a_{k_m, l_n, h_2} \end{array}, \dots,$$

$$\left. \begin{array}{c|cccc} h_f & l_1 & \cdots & l_j & \cdots & l_n \\ \hline k_1 & a_{k_1, l_1, h_f} & \vdots & a_{k_1, l_j, h_f} & \cdots & a_{k_1, l_n, h_f} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_m & a_{k_m, l_1, h_f} & \cdots & a_{k_m, l_j, h_f} & \cdots & a_{k_m, l_n, h_f} \end{array} \right\},$$

where $K = \{k_1, k_2, \dots, k_m\}$, $L = \{l_1, l_2, \dots, l_n\}$, $H = \{h_1, h_2, \dots, h_f\}$, and for $1 \leq i \leq m$, $1 \leq j \leq n$, $1 \leq g \leq f : a_{k_i, l_j, h_g} \in \mathcal{X}$.

3.2 Operations over 3-DIMs

As we saw in Section 1.5, there are 2 ($= 2!$) IMs, related to the operation “transposition”: the standard IM and its transposed IM. For 3-DIMs, we will see below that there are 6 ($= 3!$) cases: the standard 3-DIM and five different transposed 3-DIMs.

First, we will discuss the case $n = 3$ and after this, the general case. The geometrical and analytical forms of the separate transposed 3DIMs are the following.

Let $[1, 2, 3]$ be the standard permutation (identity) corresponds to the standard 3-DIM, i.e.,

$$\left(\begin{array}{c} H \\ K \left| \begin{array}{c} \diagup \\ L \end{array} \right. \end{array} \right)^{[1,2,3]} = \begin{array}{c} H \\ K \left| \begin{array}{c} \diagup \\ L \end{array} \right. \end{array}$$

The remaining permutations will correspond to the respective transpositions, as follows:

$$[K, L, H, \{a_{k_i, l_j, h_g}\}]^{[1,2,3]} = [K, L, H, \{a_{k_i, l_j, h_g}\}].$$

[1, 3, 2]-transposition

$$\left(\begin{array}{c} H \\ K \left| \begin{array}{c} \diagup \\ L \end{array} \right. \end{array} \right)^{[1,3,2]} = \begin{array}{c} L \\ K \left| \begin{array}{c} \diagup \\ H \end{array} \right. \end{array}$$

$$[K, L, H, \{a_{k_i, l_j, h_g}\}]^{[1,3,2]} = [K, H, L, \{a_{k_i, h_g, l_j}\}];$$

[2, 1, 3]-transposition

$$\left(\begin{array}{c} H \\ \diagup \\ K \mid L \end{array} \right)^{[2,1,3]} = \begin{array}{c} H \\ \diagup \\ L \mid K \end{array}$$

$$[K, L, H, \{a_{k_i, l_j, h_g}\}]^{[2,1,3]} = [L, K, H, \{a_{l_j, k_i, h_g}\}];$$

[2, 3, 1]-transposition

$$\left(\begin{array}{c} H \\ \diagup \\ K \mid L \end{array} \right)^{[2,3,1]} = \begin{array}{c} K \\ \diagup \\ L \mid H \end{array}$$

$$[K, L, H, \{a_{k_i, l_j, h_g}\}]^{[2,3,1]} = [L, H, K, \{a_{l_j, h_g, k_i}\}];$$

[3, 1, 2]-transposition

$$\left(\begin{array}{c} H \\ \diagup \\ K \mid L \end{array} \right)^{[3,1,2]} = \begin{array}{c} L \\ \diagup \\ H \mid K \end{array}$$

$$[K, L, H, \{a_{k_i, l_j, h_g}\}]^{[3,1,2]} = [H, K, L, \{a_{h_g, k_i, l_j}\}];$$

[3, 2, 1]-transposition

$$\left(\begin{array}{c} H \\ \diagup \\ K \mid L \end{array} \right)^{[3,2,1]} = \begin{array}{c} K \\ \diagup \\ H \mid L \end{array}$$

$$[K, L, H, \{a_{k_i, l_j, h_g}\}]^{[3,2,1]} = [H, L, K, \{a_{h_g, l_j, k_i}\}].$$

For the 3-DIMs $A = [K, L, H, \{a_{k_i, l_j, h_g}\}]$, $B = [P, Q, R, \{b_{p_r, q_s, e_d}\}]$, operations that are analogous of the usual matrix operations of addition and multiplication are defined, as well as other, specific ones.

Addition

$$A \oplus_{(\circ)} B = [K \cup P, L \cup Q, H \cup R, \{c_{t_u, v_w, x_y}\}],$$

where

$$c_{t_u, v_w, x_y} = \begin{cases} a_{k_i, l_j, h_g}, & \text{if } t_u = k_i \in K, v_w = l_j \in L \text{ and } x_y = h_g \in H - R \\ & \text{or } t_u = k_i \in K, v_w = l_j \in L - Q \text{ and } x_y = h_g \in H \\ & \text{or } t_u = k_i \in K - P, v_w = l_j \in L \text{ and } x_y = h_g \in H \\ b_{p_r, q_s, e_d}, & \text{if } t_u = p_r \in P, v_w = q_s \in Q \text{ and } x_y = e_d \in R - H \\ & \text{or } t_u = p_r \in P, v_w = q_s \in Q - L \text{ and } x_y = e_d \in R \\ & \text{or } t_u = p_r \in P - K, v_w = q_s \in Q \text{ and } x_y = e_d \in R \\ a_{k_i, l_j, h_g} \circ b_{p_r, q_s, e_d}, & \text{if } t_u = k_i = p_r \in K \cap P, v_w = l_j = q_s \in L \cap Q \\ & \text{and } x_y = h_g = e_d \in H \cap R \\ e_0, & \text{otherwise} \end{cases}$$

Termwise multiplication

$$A \otimes_{(\circ)} B = [K \cap P, L \cap Q, H \cap R, \{c_{t_u, v_w, x_y}\}],$$

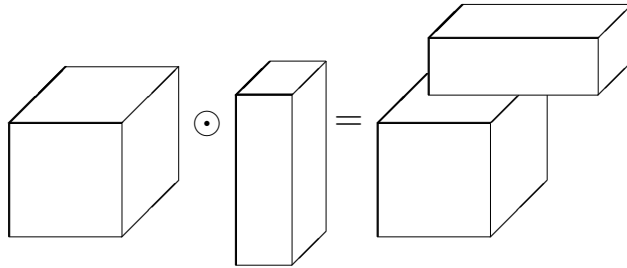
where

$$c_{t_u, v_w, x_y} = a_{k_i, l_j, h_g} \circ b_{p_r, q_s, e_d},$$

for $t_u = k_i = p_r \in K \cap P, v_w = l_j = q_s \in L \cap Q$ and $x_y = h_g = e_d \in H \cap R$.

Multiplication

This operation is related to the operation “transposition”. There are six different operations “multiplication”. Geometrically, the first (standard) multiplication has the following form



Above, we kept the previous notations for the components of 3-DIMs A and B by analogy with their 2-dimensional alternatives. Below we will start using another notation for the components of these IMs, corresponding to the general case when $n \geq 3$.

Let $[\alpha, \beta, \gamma]$ be a permutation of the ordered triple $[1, 2, 3]$. In [20], there are five different operations “multiplication”. Let the 3-DEIMs

$$\begin{aligned} A &= [K_1, K_2, K_3, \{a_{k_1, s_1}, k_{2, s_2}, k_{3, s_3}\}] \\ B &= [L_1, L_2, L_3, \{b_{l_1, t_1}, l_{2, t_2}, l_{3, t_3}\}] \end{aligned}$$

be given.

Operation $[\alpha, \beta, \gamma]$ -multiplication is defined in [20] by:

$$A \odot_{(\circ, *)}^{[\alpha, \beta, \gamma]} B = A \odot_{(\circ, *)} B^{[\alpha, \beta, \gamma]},$$

where the **first (standard) multiplication operation** $\odot_{(\circ, *)}$ is:

$$A \odot_{(\circ, *)}^1 B = [K_1 \cup (L_1 - K_2), L_2 \cup (K_2 - L_1), K_3 \cup L_3, \{c_{p_1, q_1}, p_{2, q_2}, p_{3, q_3}\}],$$

where

$$c_{p_1, q_1}, p_{2, q_2}, p_{3, q_3} = \begin{cases} a_{k_1, s_1}, k_{2, s_2}, k_{3, s_3}, & \begin{aligned} &\text{if } p_{1, q_1} = k_{1, s_1} \in K_1 \\ &\text{and } p_{2, q_2} = k_{2, s_2} \in K_2 - L_1 - L_2 \\ &\text{and } p_{3, q_3} = k_{3, s_3} \in K_3 \\ &\text{or } p_{1, q_1} = k_{1, s_1} \in K_1 - L_1 - L_2 \\ &\text{and } p_{2, q_2} = k_{2, s_2} \in K_2 \\ &\text{and } p_{3, q_3} = k_{3, s_3} \in K_3; \end{aligned} \\ b_{l_1, t_1}, l_{2, t_2}, l_{3, t_3}, & \begin{aligned} &\text{if } p_{1, q_1} = l_{1, t_1} \in L_1 \\ &\text{and } p_{2, q_2} = l_{2, t_2} \in L_2 - K_1 - K_2 \\ &\text{and } p_{3, q_3} = l_{3, t_3} \in L_3 \\ &\text{or } p_{1, q_1} = l_{1, t_1} \in L_1 - K_1 - K_2 \\ &\text{and } p_{2, q_2} = l_{2, t_2} \in L_2 \\ &\text{and } p_{3, q_3} = l_{3, t_3} \in L_3; \end{aligned} \\ \overset{\circ}{\underset{\in K_2 \cap L_1}{k_{2, s_2} = l_{1, t_1}}}(a_{k_1, s_1}, k_{2, s_2}, k_{3, s_3} * b_{l_1, t_1}, l_{2, t_2}, l_{3, t_3}) & \begin{aligned} &\text{if } p_{1, q_1} = k_{1, s_1} \in K_1 \\ &\text{and } p_{2, q_2} = l_{2, t_2} \in L_2 \\ &\text{and } p_{3, q_3} = k_{3, s_3} = l_{3, t_3} \in K_3 \cap L_3; \end{aligned} \\ e_0, & \text{otherwise.} \end{cases}$$

Now, following [29], we introduce two other operations “multiplication”.

The second operation “multiplication”

Let $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in \{1, 2, 3\}, \alpha \neq \beta \neq \gamma \neq \alpha, \delta \neq \varepsilon \neq \zeta \neq \delta$. Then

$$A \odot_{(\circ, *, \alpha, \zeta)}^2 B = [K_\alpha, L_\zeta, \{c_{p_1, q_1}, p_{2, q_2}\}],$$

where

$$c_{p_1, q_1, p_2, q_2} = \begin{cases} \circ (a_{k_1, s_1, k_2, s_2, k_3, s_3} * b_{l_1, t_1, l_2, t_2, l_3, t_3}), & \text{if } p_1, q_1 = k_{\alpha, s_{\alpha}} \in K_{\alpha} \\ & \text{and } p_2, q_2 = l_{\varepsilon, t_{\varepsilon}} \in L_{\varepsilon}; \\ e_0, & \text{otherwise.} \end{cases}$$

Therefore, this operation generates a 2-DEIM, i.e., decreases the number of dimensions.

The numbers α and ζ in the record $\odot_{(\circ, *, \alpha, \zeta)}$ mean that the α -component of the first 3-DEIM and the ζ -component of the second 3-DEIM are used as the two components of the new 2-DEIM. These two components are interpreted as the first and last components of the new 2-DEIM.

Third operation “multiplication”

Let $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in \{1, 2, 3\}, \alpha \neq \beta \neq \gamma \neq \alpha, \delta \neq \varepsilon \neq \zeta \neq \delta$. Then

$$A \odot_{(\circ, *, \gamma, \delta)}^3 B = [K_{\alpha}, K_{\beta}, L_{\varepsilon}, L_{\zeta}, \{c_{p_1, q_1, p_2, q_2, p_3, q_3, p_4, q_4}\}],$$

where

$$c_{p_1, q_1, p_2, q_2, p_3, q_3, p_4, q_4} = \begin{cases} \circ (a_{k_1, s_1, k_2, s_2, k_3, s_3} * b_{l_1, t_1, l_2, t_2, l_3, t_3}), & \text{if } p_1, q_1 = k_{\alpha, s_{\alpha}} \in K_{\alpha} \\ & \text{and } p_2, q_2 = k_{\beta, s_{\beta}} \in K_{\beta} \\ & \text{and } p_3, q_3 = l_{\varepsilon, t_{\varepsilon}} \in L_{\varepsilon} \\ & \text{and } p_4, q_4 = l_{\zeta, t_{\zeta}} \in L_{\zeta} \\ e_0, & \text{otherwise} \end{cases}$$

Therefore, this operation generates a 4-DEIM, i.e., it increases the number of dimensions.

Numbers γ and δ in the record $\odot_{(\circ, *, \gamma, \delta)}^3$ mean that the two components α and β of the first 3-DEIM and the two components ε and ζ of the second 3-DEIM are used as the four components of the resultant 4-DEIM.

Structural subtraction

$$A \ominus B = [K - P, L - Q, H - R, \{c_{t_u, v_w, x_y}\}],$$

where “ $-$ ” is the set-theoretic operation “subtraction” and $c_{t_u, v_w, x_y} = a_{k_i, l_j, h_g}$, for $t_u = k_i \in K - P, v_w = l_j \in L - Q$ and $x_y = h_g \in H - R$.

Operation $\circ$ with a constant

$$\alpha \circ A = [K, L, H, \{\alpha \circ a_{k_i, l_j, h_g}\}],$$

where α is a constant and $\circ$ – a suitable operation, e.g., $\circ \in \{+, -, \cdot, : \}$.

3.3 Definition of an n -Dimensional EIM

An n -Dimensional EIM (n -DEIM) with index sets $K_1, K_2, \dots, K_n$ ($K_1, K_2, \dots, K_n \subseteq \mathcal{I}^*$) and elements from set $\mathcal{X}$ is called the object:

$$A = [K_1, K_2, \dots, K_n, \{a_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}}\}]$$

where $K_i = \{k_{i,1}, k_{i,2}, \dots, k_{i,m_i}\}$, $m_i \geq 1$ and $a_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}} \in \mathcal{X}$ for $1 \leq i \leq n$ and $1 \leq s_i \leq m_i$. It is introduced in [29].

The n -DEIM A has the following $\frac{n(n-1)}{2}$ different representations as a 2-DIM:

$$\begin{aligned}
 A &= \left\{ \begin{array}{c|ccc} \langle k_{3,s_3}, \dots, k_{n,s_n} \rangle & k_{2,1} & \dots & k_{2,m_2} \\ \hline k_{1,1} & a_{k_{1,1}, k_{2,1}, k_{3,s_3}, \dots, k_{n,s_n}} & \dots & a_{k_{1,1}, k_{2,m_2}, k_{3,s_3}, \dots, k_{n,s_n}} \\ \vdots & \vdots & \ddots & \vdots \\ k_{1,i} & a_{k_{1,i}, k_{2,1}, k_{3,s_3}, \dots, k_{n,s_n}} & \dots & a_{k_{1,i}, k_{2,m_2}, k_{3,s_3}, \dots, k_{n,s_n}} \\ \vdots & \vdots & \ddots & \vdots \\ k_{1,m_1} & a_{k_{1,m_1}, k_{2,1}, k_{3,s_3}, \dots, k_{n,s_n}} & \dots & a_{k_{1,m_1}, k_{2,m_2}, k_{3,s_3}, \dots, k_{n,s_n}} \end{array} \right. \\
 &= \dots = \\
 &= \left\{ \begin{array}{c|ccc} \langle k_{1,s_1}, \dots, k_{n-2,s_{n-2}} \rangle & k_{n,1} & \dots & \\ \hline k_{n-1,1} & a_{k_{1,s_1}, \dots, k_{n-2,s_{n-2}}, k_{n-1,1}, k_{n,1}} & \dots & \\ \vdots & \vdots & \ddots & \\ k_{n-1,i} & a_{k_{1,s_1}, \dots, k_{n-2,s_{n-2}}, k_{n-1,i}, k_{n,1}} & \dots & \\ \vdots & \vdots & \ddots & \\ k_{n-1,m_{n-1}} & a_{k_{1,s_1}, \dots, k_{n-2,s_{n-2}}, k_{n-1,m_{n-1}}, k_{n,1}} & \dots & \\ \hline \dots & k_{n,m_n} & \dots & \\ \dots & a_{k_{1,s_1}, \dots, k_{n-2,s_{n-2}}, k_{n-1,1}, k_{n,m_n}} & \dots & \\ \ddots & \vdots & \ddots & \\ \dots & a_{k_{1,s_1}, \dots, k_{n-2,s_{n-2}}, k_{n-1,i}, k_{n,m_n}} & \dots & \\ \ddots & \vdots & \ddots & \\ \dots & a_{k_{1,s_1}, \dots, k_{n-2,s_{n-2}}, k_{n-1,m_{n-1}}, k_{n,m_n}} & \dots & \end{array} \right. \\
 &= \left\{ \langle k_{1,s_1}, \dots, k_{n-2,s_{n-2}} \rangle \in K_1 \times \dots \times K_{n-2} \right\}.
 \end{aligned}$$

3.4 Operations over n -DEIMs

Here, we introduce the basic operations over n -DEIMs, by analogy with [29, 37] and with the previous sections.

Let us have two n -DEIMs A from section 3.3 and

$$B = [L_1, L_2, \dots, L_n, \{b_{l_{1,t_1}, l_{2,t_2}, \dots, l_{n,t_n}}\}].$$

Let everywhere below i be a natural number for which $1 \leq i \leq n$. Following [29], we introduce the following operations over A and B .

Operation “Addition”

$$A \oplus_{(\circ)} B = [K_1 \cup L_1, K_2 \cup L_2, \dots, K_n \cup L_n, \{c_{p_{1,q_1}, p_{2,q_2}, \dots, p_{n,q_n}}\}],$$

where

$$c_{p_{1,q_1}, p_{2,q_2}, \dots, p_{n,q_n}} = \begin{cases} a_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}}, & \text{if for each } i: p_{i,q_i} = k_{i,s_i} \in K_i \\ & \text{and there exists } i \text{ such that } k_{i,s_i} \notin L_i; \\ b_{l_{1,t_1}, l_{2,t_2}, \dots, l_{n,t_n}}, & \text{if for each } i: p_{i,q_i} = l_{i,t_i} \in K_i \\ & \text{and there exists } i \text{ such that } l_{i,t_i} \notin K_i; \\ a_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}} \circ b_{l_{1,t_1}, l_{2,t_2}, \dots, l_{n,t_n}}, & \text{if for each } i: p_{i,q_i} = k_{i,s_i} = l_{i,t_i} \\ & \in K_i \cap L_i; \\ e_{\circ}, & \text{otherwise} \end{cases}$$

Operation “Termwise multiplication”

$$A \otimes_{(\circ)} B = [K_1 \cap L_1, K_2 \cap L_2, \dots, K_n \cap L_n, \{c_{p_{1,q_1}, p_{2,q_2}, \dots, p_{n,q_n}}\}],$$

where

$$c_{p_{1,q_1}, p_{2,q_2}, \dots, p_{n,q_n}} = a_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}} \circ b_{l_{1,t_1}, l_{2,t_2}, \dots, l_{n,t_n}}$$

for each i , $p_{i,q_i} = k_{i,s_i} = l_{i,t_i} \in K_i \cap L_i$.

Operation “Multiplication”

Here we will discuss only one of the possible cases for this operation.

Let A and B be the two n -DEIM from Section 1.3. Then we define one of the possible general cases for operation multiplication. It includes the first, third and the partial case of the second new operations introduced in Section 3.2. By the moment, an **Open Problem** which is the general form for the second multiplication.

Let us define the symbol $\exists_{1 \div n} i$ that means “there is at least one and no more than $n - 1$ values of i ” that we will use in the next definition and in some definitions in the next chapters.

Let c and d are two natural numbers for which $1 \leq c \leq n-1$, $0 \leq d \leq n-2$ and $c+d \leq n-1$. Then we obtain the following $(2c+d)$ -DEIM

$$A \odot_{(\circ, *, c, d)} B = [K_1, \dots, K_c, K_{n-d+1} \cup L_1, \dots, K_n \cup L_d, L_{n-c+1}, \dots, L_n, \\ \{c_{p_1, q_1}, p_2, q_2, \dots, p_{2c+d}, q_{2c+d}\}],$$

where

$$\begin{aligned} & c_{p_1, q_1}, p_2, q_2, \dots, p_{2c+d}, q_{2c+d} \\ & = \left\{ \begin{array}{ll} a_{k_1, s_1}, k_2, s_2, \dots, k_n, s_n, & \text{if } (\forall i)(k_i \in K_i) \ \& \ (\exists_{1 \div n} j)(k_j \notin \bigcup_{i=1}^n L_j) \\ b_{l_1, t_1}, l_2, t_2, \dots, l_n, t_n, & \text{if } (\forall i)(l_i \in L_i) \ \& \ (\exists_{1 \div n} j)(l_j \notin \bigcup_{i=1}^n K_j) \\ \circ \quad (a_{k_1, s_1}, k_2, s_2, \dots, k_n, s_n * b_{l_1, t_1}, l_2, t_2, \dots, l_n, t_n), & \\ \begin{array}{ll} k_{c+1}, s_{c+1} = l_{d+1}, t_{d+1} \in K_{c+1} \cap L_{d+1} & \text{if } p_{1, q_1} = k_{1, s_1} \in K_1, \dots, p_{c, q_c} = k_{c, s_c} \in K_c \\ k_{n-d}, s_{n-d} = l_{n-c}, t_{n-c} \in K_{n-d} \cap L_{n-c} & p_{c+1}, q_{c+1} = k_{n-d+1}, s_{n-d+1} \\ & = l_{1, t_1} \in K_{n-d+1} \cap L_1 \\ & \vdots \\ & p_{c+d}, q_{c+d} = k_{n, s_n} = l_{d, t_d} \in K_n \cap L_d \\ & p_{c+d+1}, q_{c+d+1} = l_{n-c+1}, t_{n-c+1} \in L_{n-c+1} \\ & \vdots \\ & p_{2c+d}, q_{2c+d} = l_{n, t_n} \in L_n; \end{array} \\ e_0, & \text{otherwise.} \end{array} \right. \end{aligned}$$

Operation “Structural subtraction”

$$A \ominus B = [K_1 - L_1, K_2 - L_2, \dots, K_n - L_n, \{c_{p_1, q_1}, p_2, q_2, \dots, p_n, q_n\}],$$

where

$$c_{p_1, q_1}, p_2, q_2, \dots, p_n, q_n = a_{k_1, s_1}, k_2, s_2, \dots, k_n, s_n$$

for each i , $p_{i, q_i} = k_{i, s_i} \in K_i - L_i$.

Operation $\circ$ with a constant

$$\alpha \circ A = [K_1, K_2, \dots, K_n, \{\alpha \circ a_{k_1, s_1}, k_2, s_2, \dots, k_n, s_n\}],$$

where α , as in the previous chapters, is a constant and $\circ$ – a suitable operation depending on the type of A

Operation “Projection”

Let $L_i \subseteq K_i$. Then,

$$\text{pr}_{L_1, L_2, \dots, L_n} A = [L_1, L_2, \dots, L_n, \{b_{l_1, t_1}, l_2, t_2, \dots, l_n, t_n\}],$$

where

$$b_{l_{1,t_1}, l_{2,t_2}, \dots, l_{n,t_n}} = a_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}}$$

for each i for which $l_{i,t_i} = k_{i,s_i} \in L_i$.

Operations “Reduction” over an n -DEIM

First, we introduce operation $(\perp, \dots, \perp, k_{i,s_i}, \perp, \dots, \perp)$ -reduction of a given n -DEIM A by

$$\begin{aligned} & A_{(\perp, \dots, \perp, k_{i,s_i}, \perp, \dots, \perp)} \\ &= [K_1, \dots, K_{i-1}, K_i - \{k_{i,s_i}\}, K_{i+1}, \dots, K_n, \{c_{p_{1,q_1}, p_{2,q_2}, \dots, p_{n,q_n}}\}] \end{aligned}$$

where

$$c_{p_{1,q_1}, p_{2,q_2}, \dots, p_{n,q_n}} = a_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}}$$

and $p_{j,q_j} = k_{j,s_j} \in K_j$ for $1 \leq j \leq n$ and $j \neq i$; and $p_{i,r} = k_{i,s_i} \in K_i$ for $1 \leq r \leq m_i$ and $r \neq s_i$.

Second, we define

$$\begin{aligned} A_{(k_{1,p_1}, k_{2,p_2}, \dots, k_{n,p_n})} &= (\dots (A_{(k_{1,p_1}, \perp, \dots, \perp)})(\perp, k_{2,p_2}, \perp, \dots, \perp) \dots)(\perp, \dots, \perp, k_{n,p_n}) \\ &= [K_1 - \{k_{1,p_1}\}, K_2 - \{k_{2,p_2}\}, \dots, K_n - \{k_{n,p_n}\}, \{c_{p_{1,q_1}, p_{2,q_2}, \dots, p_{n,q_n}}\}], \end{aligned}$$

where

$$c_{p_{1,q_1}, p_{2,q_2}, \dots, p_{n,q_n}} = a_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}}$$

and for each i : $p_{i,q_i} = k_{i,s_i} \in K_i - \{k_{i,p_i}\}$ and $1 \leq p_{i,q_i} \leq m_i$.

Third, let for each i : $L_i = \{l_{i,q_1}, l_{i,q_2}, \dots, l_{i,q_{r_i}}\} \subseteq K_i$, where $1 \leq r_i \leq m_i$. Now, we define the following operations:

$$\begin{aligned} & A_{(\perp, \dots, \perp, L_i, \perp, \dots, \perp)} \\ &= (\dots ((A_{(\perp, \dots, \perp, l_{i,q_1}, \perp, \dots, \perp)})(\perp, \dots, \perp, l_{i,q_2}, \perp, \dots, \perp) \dots)(\perp, \dots, \perp, l_{i,q_{r_i}}, \perp, \dots, \perp), \\ & A_{(L_1, L_2, \dots, L_n)} = (\dots ((A_{(L_1, \perp, \dots, \perp)})(\perp, L_2, \perp, \dots, \perp) \dots)(\perp, \dots, \perp, L_n). \end{aligned}$$

Operations “Substitution”

Let the n -DEIM A be given.

First, local substitution of an index of A is defined for the pairs of indices¹ (l_{1,p_1}, k_{1,p_1}) and/or (l_{2,p_2}, k_{2,p_2}) and/or $\dots$ and/or (l_{n,p_n}, k_{n,p_n}) , respectively, for arbitrary indices p_i ($1 \leq p_i \leq m_i$), by

$$\begin{aligned} \left[1; \frac{l_{1,p_1}}{k_{1,p_1}}\right] A &= \left[\frac{l_{1,p_1}}{k_{1,p_1}}; \perp; \dots; \perp\right] A \\ &= [(K_1 - \{k_{1,p_1}\}) \cup \{l_{1,p_1}\}, K_2, \dots, K_n, \{a_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}}\}]; \end{aligned}$$

¹ Here we give two different notations – a new one and the older notation, used in [20].

$$\begin{aligned}
\left[2; \frac{l_{2,p_2}}{k_{2,p_2}}\right] A &= \left[\perp; \frac{l_{2,p_2}}{k_{2,p_2}}; \perp; \dots; \perp\right] A \\
&= [K_1, (K_2 - \{k_{2,p_2}\}) \cup \{l_{2,p_2}\}, K_3, \dots, K_n, \{a_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}}\}] ; \\
&\vdots \\
\left[n; \frac{l_{n,p_n}}{k_{n,p_n}}\right] A &= \left[\perp; \dots; \perp; \frac{l_{n,p_n}}{k_{n,p_n}}\right] A \\
&= [K_1, K_2, \dots, K_{n-1}, K_n - \{k_{n,p_n}\}) \cup \{l_{n,p_n}\}, \{a_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}}\}] .
\end{aligned}$$

Second,

$$\begin{aligned}
\left[\frac{l_{1,p_1}}{k_{1,p_1}}, \frac{l_{2,p_2}}{k_{2,p_2}}, \dots, \frac{l_{n,p_n}}{k_{n,p_n}}\right] A &= \left[1; \frac{l_1}{k_1}\right] \left[2; \frac{l_2}{k_2}\right] \dots \left[n; \frac{l_n}{k_n}\right] A \\
&= \left[\frac{l_1}{k_1}; \perp; \dots; \perp\right] \left[\perp; \frac{l_2}{k_2}; \perp; \dots; \perp\right] \dots \left[\perp; \perp; \dots; \perp; \frac{l_n}{k_n}\right] A \\
&= [(K_1 - \{k_{1,p_1}\}) \cup \{l_{1,p_1}\}, (K_2 - \{k_{2,p_2}\}) \cup \{l_{2,p_2}\}, \dots, \\
&\quad (K_n - \{k_{n,p_n}\}) \cup \{l_{n,p_n}\}, \{a_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}}\}].
\end{aligned}$$

Third, let the sets of indices $L_1, L_2, \dots, L_n$ be given, where $\text{card}(L_i) = \text{card}(K_i)$. For them we define sequentially:

$$\begin{aligned}
\left[i; \frac{L_i}{K_i}\right] A &= \left[\perp; \dots; \perp; \frac{L_i}{K_i}; \perp; \dots; \perp\right] A \\
&= \left[i; \frac{l_{i,1}}{k_{i,1}} \frac{l_{i,2}}{k_{i,2}} \dots \frac{l_{i,m_i}}{k_{i,m_i}}\right] A \\
&= \left[\perp; \dots; \perp; \frac{l_{i,1}}{k_{i,1}} \frac{l_{i,2}}{k_{i,2}} \dots \frac{l_{i,m_i}}{k_{i,m_i}}; \perp; \perp\right] A, \\
\left[\frac{L_1}{K_1}, \frac{L_2}{K_2}, \dots, \frac{L_n}{K_n}\right] A &= \left[\frac{L_1}{K_1}; \perp; \dots; \perp\right] \left[\perp; \frac{L_2}{K_2}; \perp; \dots; \perp\right] \dots \left[\perp; \dots; \perp; \frac{L_n}{K_n}\right] A.
\end{aligned}$$

Fourth, let for brevity $1 \leq i < j \leq n$. Then the local substitution of a value of A is defined by:

$$\begin{aligned}
&\left\langle i, j; k_{i,p_i}, k_{j,q_j}; \frac{b_{k_{i,p_i}, k_{j,q_j}}}{a_{k_{i,p_i}, l_{j,q_j}}} \right\rangle \\
&\left\{ \begin{array}{c|ccc} \langle k_{1,s_1}, \dots, k_{i-1,p_{i-1}}, k_{i+1,p_{i+1}}, \dots, & & & \\ k_{j-1,p_{j-1}}, k_{j+1,p_{j+1}}, \dots, k_{n,s_n} \rangle & \dots & k_{j,q_j} & \dots \\ \vdots & \ddots & \vdots & \ddots \\ k_{i,p_i} & \dots & a_{k_{i,p_i}, k_{j,q_j}} & \dots \\ \vdots & \ddots & \vdots & \ddots \end{array} \right. \\
&| \langle k_{1,s_1}, \dots, k_{i-1,p_{i-1}}, k_{i+1,p_{i+1}}, \dots, k_{j-1,p_{j-1}}, k_{j+1,p_{j+1}}, \dots, k_{n,s_n} \rangle \\
&\in K_1 \times \dots \times K_{i-1} \times K_{i+1} \times \dots \times K_{j-1} \times K_{j+1} \times \dots \times K_n \}
\end{aligned}$$

$$= \left\{ \begin{array}{c|ccc} \langle k_{1,s_1}, \dots, k_{i-1,p_{i-1}}, k_{i+1,p_{i+1}}, \dots, \\ k_{j-1,p_{j-1}}, k_{j+1,p_{j+1}}, \dots, k_{n,s_n} \rangle & \dots & k_{j,q_j 2} & \dots \\ \vdots & \ddots & \vdots & \ddots \\ k_{i,p_i} & \dots & b_{k_{i,p_i}, k_{j,q_j}} & \dots \\ \vdots & \ddots & \vdots & \ddots \end{array} \right. \\ \left. \begin{array}{l} | \langle k_{1,s_1}, \dots, k_{i-1,p_{i-1}}, k_{i+1,p_{i+1}}, \dots, k_{j-1,p_{j-1}}, k_{j+1,p_{j+1}}, \dots, k_{n,s_n} \rangle \\ \in K_1 \times \dots \times K_{i-1} \times K_{i+1} \times \dots \times K_{j-1} \times K_{j+1} \times \dots \times K_n \end{array} \right\}.$$

3.5 Aggregation operations over n -DEIM

Let the n -DEIM A be given and let $l_i \notin K_i$ for i ($1 \leq i \leq n$) be n indices. First, by analogy with [48], we introduce the following four aggregation operations over A .

Max-aggregation

$$\mathbf{agg}_{(\max)}(A, i, l) =$$

$$\begin{array}{c|ccc} & k_{i,1} & & \dots \\ \hline l & \max & a_{k_{1,s_1}, \dots, k_{i-1,s_{i-1}}, k_{i,1}, k_{i+1,s_{i+1}}, \dots, k_{n,s_n}} & \dots \\ & \langle k_{1,s_1}, \dots, k_{i-1,s_{i-1}}, k_{i+1,s_{i+1}}, \dots, k_{n,s_n} \rangle & & \\ & \in K_1 \times \dots \times K_{i-1} \times K_{i+1} \times \dots \times K_n & & \\ \hline & k_{i,m_i} & & \\ \dots & \max & a_{k_{1,s_1}, \dots, k_{i-1,s_{i-1}}, k_{i,m_i}, k_{i+1,s_{i+1}}, \dots, k_{n,s_n}} & \\ \dots & \langle k_{1,s_1}, \dots, k_{i-1,s_{i-1}}, k_{i+1,s_{i+1}}, \dots, k_{n,s_n} \rangle & & \\ & \in K_1 \times \dots \times K_{i-1} \times K_{i+1} \times \dots \times K_n & & \end{array}$$

Min-aggregation

$$\mathbf{agg}_{(\min)}(A, i, l) =$$

$$\begin{array}{c|ccc} & k_{i,1} & & \dots \\ \hline l & \min & a_{k_{1,s_1}, \dots, k_{i-1,s_{i-1}}, k_{i,1}, k_{i+1,s_{i+1}}, \dots, k_{n,s_n}} & \dots \\ & \langle k_{1,s_1}, \dots, k_{i-1,s_{i-1}}, k_{i+1,s_{i+1}}, \dots, k_{n,s_n} \rangle & & \\ & \in K_1 \times \dots \times K_{i-1} \times K_{i+1} \times \dots \times K_n & & \\ \hline & k_{i,m_i} & & \\ \dots & \min & a_{k_{1,s_1}, \dots, k_{i-1,s_{i-1}}, k_{i,m_i}, k_{i+1,s_{i+1}}, \dots, k_{n,s_n}} & \\ \dots & \langle k_{1,s_1}, \dots, k_{i-1,s_{i-1}}, k_{i+1,s_{i+1}}, \dots, k_{n,s_n} \rangle & & \\ & \in K_1 \times \dots \times K_{i-1} \times K_{i+1} \times \dots \times K_n & & \end{array}$$

Sum-aggregation

$$\mathbf{agg}_{(\text{sum})}(A, i, l) =$$

$$\begin{array}{c|c}
 l & \begin{array}{c} k_{i,1} \quad \dots \\ \hline \sum a_{k_{1,s_1}, \dots, k_{i-1,s_{i-1}}, k_{i,1}, k_{i+1,s_{i+1}}, \dots, k_{n,s_n}} \quad \dots \\ \langle k_{1,s_1}, \dots, k_{i-1,s_{i-1}}, k_{i+1,s_{i+1}}, \dots, k_{n,s_n} \rangle \\ \in K_1 \times \dots \times K_{i-1} \times K_{i+1} \times \dots \times K_n \end{array} \\
 \hline
 \dots & \begin{array}{c} k_{i,m_i} \\ \hline \sum a_{k_{1,s_1}, \dots, k_{i-1,s_{i-1}}, k_{i,m_i}, k_{i+1,s_{i+1}}, \dots, k_{n,s_n}} \\ \langle k_{1,s_1}, \dots, k_{i-1,s_{i-1}}, k_{i+1,s_{i+1}}, \dots, k_{n,s_n} \rangle \\ \in K_1 \times \dots \times K_{i-1} \times K_{i+1} \times \dots \times K_n \end{array} \\
 \dots & \dots
 \end{array}$$

Average-aggregation

$$\mathbf{agg}_{(\text{ave})}(A, i, l) =$$

$$\begin{array}{c|c}
 l & \begin{array}{c} k_{i,1} \quad \dots \\ \hline \frac{1}{M_i} \sum a_{k_{1,s_1}, \dots, k_{i-1,s_{i-1}}, k_{i,1}, k_{i+1,s_{i+1}}, \dots, k_{n,s_n}} \quad \dots \\ \langle k_{1,s_1}, \dots, k_{i-1,s_{i-1}}, k_{i+1,s_{i+1}}, \dots, k_{n,s_n} \rangle \\ \in K_1 \times \dots \times K_{i-1} \times K_{i+1} \times \dots \times K_n \end{array} \\
 \hline
 \dots & \begin{array}{c} k_{i,m_i} \\ \hline \frac{1}{M_i} \sum a_{k_{1,s_1}, \dots, k_{i-1,s_{i-1}}, k_{i,m_i}, k_{i+1,s_{i+1}}, \dots, k_{n,s_n}} \\ \langle k_{1,s_1}, \dots, k_{i-1,s_{i-1}}, k_{i+1,s_{i+1}}, \dots, k_{n,s_n} \rangle \\ \in K_1 \times \dots \times K_{i-1} \times K_{i+1} \times \dots \times K_n \end{array} \\
 \dots & \dots
 \end{array}$$

where

$$M_i = \prod_{j=1, j \neq i}^n m_j.$$

We can immediately see that for every IM A , for every pair of indices i and j and for every $\circ \in \{\max, \min, \text{sum}, \text{ave}\}$:

$$\mathbf{agg}_{(\circ)}(\mathbf{agg}_{(\circ)}(A, j, l_1), i, l_2) = \mathbf{agg}_{(\circ)}(\mathbf{agg}_{(\circ)}(A, j, l_2), i, l_1).$$

In the case of $(0, 1)$ - n DEIMs, only operations $\mathbf{agg}_{\max}$ and $\mathbf{agg}_{\min}$ are possible.

When we use n -DEIM with elements propositions or predicates, then the aggregation operations are:

$\vee$ -aggregation

$$\text{agg}_{(\vee)}(A, i, l)$$

$$= l \left| \begin{array}{c} \hline k_{i,1} \quad \dots \\ \hline \bigvee a_{k_{1,s_1}, \dots, k_{i-1,s_{i-1}}, k_{i,1}, k_{i+1,s_{i+1}}, \dots, k_{n,s_n}} \quad \dots \\ \langle k_{1,s_1}, \dots, k_{i-1,s_{i-1}}, k_{i+1,s_{i+1}}, \dots, k_{n,s_n} \rangle \\ \in K_1 \times \dots \times K_{i-1} \times K_{i+1} \times \dots \times K_n \\ \hline \dots \quad k_{i,m_i} \\ \dots \quad \bigvee a_{k_{1,s_1}, \dots, k_{i-1,s_{i-1}}, k_{i,m_i}, k_{i+1,s_{i+1}}, \dots, k_{n,s_n}} \quad \dots \\ \langle k_{1,s_1}, \dots, k_{i-1,s_{i-1}}, k_{i+1,s_{i+1}}, \dots, k_{n,s_n} \rangle \\ \in K_1 \times \dots \times K_{i-1} \times K_{i+1} \times \dots \times K_n \end{array} \right| ,$$

 $\wedge$ -aggregation

$$\text{agg}_{(\wedge)}(A, i, l)$$

$$= l \left| \begin{array}{c} \hline k_{i,1} \quad \dots \\ \hline \bigwedge a_{k_{1,s_1}, \dots, k_{i-1,s_{i-1}}, k_{i,1}, k_{i+1,s_{i+1}}, \dots, k_{n,s_n}} \quad \dots \\ \langle k_{1,s_1}, \dots, k_{i-1,s_{i-1}}, k_{i+1,s_{i+1}}, \dots, k_{n,s_n} \rangle \\ \in K_1 \times \dots \times K_{i-1} \times K_{i+1} \times \dots \times K_n \\ \hline \dots \quad k_{i,m_i} \\ \dots \quad \bigwedge a_{k_{1,s_1}, \dots, k_{i-1,s_{i-1}}, k_{i,m_i}, k_{i+1,s_{i+1}}, \dots, k_{n,s_n}} \quad \dots \\ \langle k_{1,s_1}, \dots, k_{i-1,s_{i-1}}, k_{i+1,s_{i+1}}, \dots, k_{n,s_n} \rangle \\ \in K_1 \times \dots \times K_{i-1} \times K_{i+1} \times \dots \times K_n \end{array} \right| .$$

3.6 n -DEIM with function-type of elements

The concept of a (2-dimensional) IM with function-type of elements was described in details in [18, 20, 24]. Here, following [37], we introduce its n -dimensional form.

Let the set of all used functions be $\mathcal{F}$.

The research over IMs with function-type of elements develops in the following two directions:

- each function of set $\mathcal{F}$ has one argument and it is exactly x (i.e., it is not possible that one of the functions has argument x and another function has argument y) – let us mark the set of these functions by $\mathcal{F}_x^1$;
- each function of set $\mathcal{F}$ has one argument, but that argument might be different for the different functions or the different functions of set $\mathcal{F}$ have different numbers of arguments.

The n -DEIM with Function-type of Elements (n -DEIMFE) has the form

$$A = [K_1, K_2, \dots, K_n, \{f_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}}\}]$$

where $K_i = \{k_{i,1}, k_{i,2}, \dots, k_{i,m_i}\}$, $m_i \geq 1$ and $f_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}} \in \mathcal{F}$ for $1 \leq i \leq n$ and $1 \leq s_i \leq m_i$.

The other form of the n -DEIM A representations is:

$$\begin{aligned}
 A &= \left\{ \begin{array}{c|ccc} \langle k_{3,s_3}, \dots, k_{n,s_n} \rangle & k_{2,1} & \cdots & k_{2,m_2} \\ \hline k_{1,1} & f_{k_{1,1}, k_{2,1}, k_{3,s_3}, \dots, k_{n,s_n}} & \cdots & f_{k_{1,1}, k_{2,m_2}, k_{3,s_3}, \dots, k_{n,s_n}} \\ \vdots & \vdots & \ddots & \vdots \\ k_{1,i} & f_{k_{1,i}, k_{2,1}, k_{3,s_3}, \dots, k_{n,s_n}} & \cdots & f_{k_{1,i}, k_{2,m_2}, k_{3,s_3}, \dots, k_{n,s_n}} \\ \vdots & \vdots & \ddots & \vdots \\ k_{1,m_1} & f_{k_{1,m_1}, k_{2,1}, k_{3,s_3}, \dots, k_{n,s_n}} & \cdots & f_{k_{1,m_1}, k_{2,m_2}, k_{3,s_3}, \dots, k_{n,s_n}} \end{array} \right. \\
 &\quad | \langle k_{3,s_3}, \dots, k_{n,s_n} \rangle \in K_3 \times \cdots \times K_n \} \\
 &= \cdots \\
 &= \left\{ \begin{array}{c|ccc} \langle k_{1,s_1}, \dots, k_{n-2,s_{n-2}} \rangle & k_{n,1} & \cdots & k_{n,m_n} \\ \hline k_{n-1,1} & f_{k_{1,s_1}, \dots, k_{n-1,1}, k_{n,1}} & \cdots & f_{k_{1,s_1}, \dots, k_{n-1,1}, k_{n,m_n}} \\ \vdots & \vdots & \ddots & \vdots \\ k_{n-1,i} & f_{k_{1,s_1}, \dots, k_{n-1,i}, k_{n,1}} & \cdots & f_{k_{1,s_1}, \dots, k_{n-1,i}, k_{n,m_n}} \\ \vdots & \vdots & \ddots & \vdots \\ k_{n-1,m_{n-1}} & f_{k_{1,s_1}, \dots, k_{n-1,m_{n-1}}, k_{n,1}} & \cdots & f_{k_{1,s_1}, \dots, k_{n-1,m_{n-1}}, k_{n,m_n}} \end{array} \right. \\
 &\quad | \langle k_{1,s_1}, \dots, k_{n-2,s_{n-2}} \rangle \in K_1 \times \cdots \times K_{n-2} \}.
 \end{aligned}$$

The n -DEIMFE has this form independently of the form of its elements. They can be functions from $\mathcal{F}_x^1$ having one, exactly determined argument (in the present case - x), as well as functions with a lot of arguments. The set of s -argument functions will be marked by $\mathcal{F}^s$, where $s \geq 2$.

Now, we give the definitions of the operations over n -DEIMFEs. The first three of them are in more general form than in [18, 20].

Let the n -DEIMFEs

$$\begin{aligned}
 F &= [K_1, K_2, \dots, K_n, \{f_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}}\}], \\
 G &= [L_1, L_2, \dots, L_n, \{g_{l_{1,t_1}, l_{2,t_2}, \dots, l_{n,t_n}}\}],
 \end{aligned}$$

where it holds that $f_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}}, g_{l_{1,t_1}, l_{2,t_2}, \dots, l_{n,t_n}} \in \mathcal{F}_x^1$ or $f_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}}, g_{l_{1,t_1}, l_{2,t_2}, \dots, l_{n,t_n}} \in \mathcal{F}^d$ simultaneously, where $d \geq 2$.

Following [21] we introduce the operation “addition” over A and B by:

$$F \oplus_{(\circ)} G = [K_1 \cup L_1, K_2 \cup L_2, \dots, K_n \cup L_n, \{h_{p_{1,q_1}, p_{2,q_2}, \dots, p_{n,q_n}}\}],$$

where

$$h_{p_{1,q_1}, p_{2,q_2}, \dots, p_{n,q_n}} = \begin{cases} f_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}}, & \text{if for each } i: p_{i,q_i} = k_{i,s_i} \in K_i \\ & \text{and there is } i \text{ such that } k_{i,s_i} \notin L_i; \\ g_{l_{1,t_1}, l_{2,t_2}, \dots, l_{n,t_n}}, & \text{if for each } i: p_{i,q_i} = l_{i,t_i} \in K_i \\ & \text{and there is } i \text{ such that } l_{i,t_i} \notin K_i; \\ f_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}} \circ g_{l_{1,t_1}, l_{2,t_2}, \dots, l_{n,t_n}}, & \text{if for each } i: p_{i,q_i} = k_{i,s_i} = l_{i,t_i} \\ & \in K_i \cap L_i; \\ e_0, & \text{otherwise,} \end{cases}$$

where operation $\circ$ is defined in $\mathcal{F}_x^1$ or in $\mathcal{F}^d$, with respect to the form of the f - and g -functions. For example, if

$$\begin{aligned} f(x) &= x^5 - 3x^4 + x^3 + 1, \\ g(x) &= x^6 - 2x^5 - 3, \end{aligned}$$

and operation $\circ$ is $+$, then

$$h(x) = f(x) \circ g(x) = x^6 - x^5 - 3x^4 + x^3 - 2.$$

It is important to mention that here and in some next cases that we draw, in the case “otherwise” must stay the symbol “ $\perp$ ”.

Therefore, the operation addition over two n -DEIMFEs is similar to this one from Section 1.3. The same is the situation with operations “termwise multiplication”, “multiplication”, “structural subtraction”, “operation with a constant”.

The next operations are specific for the n -DEIMFEs (for the different dimension).

Let

$$\begin{aligned} A &= [K_1, K_2, \dots, K_n, \{a_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}}\}], \\ F &= [L_1, L_2, \dots, L_n, \{f_{l_{1,t_1}, l_{2,t_2}, \dots, l_{n,t_n}}\}], \end{aligned}$$

be given, where $a_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}} \in \mathbb{R}$ and $f_{l_{1,t_1}, l_{2,t_2}, \dots, l_{n,t_n}} \in \mathcal{F}_x^1$ or $a_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}} \in \mathbb{R}^d$ and $f_{l_{1,t_1}, l_{2,t_2}, \dots, l_{n,t_n}} \in \mathcal{F}^d$ simultaneously, where $d \geq 2$. Then, we can define the following operations.

Operation $\boxplus$

$$A \boxplus F = [K_1 \cup L_1, K_2 \cup L_2, \dots, K_n \cup L_n, \{h_{p_{1,q_1}, p_{2,q_2}, \dots, p_{n,q_n}}\}],$$

where

$$= \begin{cases} h_{p_{1,q_1}, p_{2,q_2}, \dots, p_{n,q_n}} & \\ a_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}}, & \text{if for each } i: p_{i,q_i} = k_{i,s_i} \in K_i \\ & \text{and there exists } i \text{ such that } k_{i,s_i} \notin L_i; \\ f_{l_{1,t_1}, l_{2,t_2}, \dots, l_{n,t_n}}, & \text{if for each } i: p_{i,q_i} = l_{i,t_i} \in K_i \\ & \text{and there exists } i \text{ such that } l_{i,t_i} \notin K_i; \\ a_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}} \circ f_{l_{1,t_1}, l_{2,t_2}, \dots, l_{n,t_n}}, & \text{if for each } i: \\ & p_{i,q_i} = k_{i,s_i} = l_{i,t_i} \in K_i \cap L_i; \\ e_o, & \text{otherwise,} \end{cases}$$

where $\circ \in \{+, -, \times, :\}$ and $a_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}} \in \mathbb{R}$.

$$F \boxplus A = [K_1 \cup L_1, K_2 \cup L_2, \dots, K_n \cup L_n, \{h_{p_{1,q_1}, p_{2,q_2}, \dots, p_{n,q_n}}\}],$$

where

$$= \begin{cases} h_{p_{1,q_1}, p_{2,q_2}, \dots, p_{n,q_n}} & \\ a_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}}, & \text{if for each } i: p_{i,q_i} = k_{i,s_i} \in K_i \\ & \text{and there exists } i \text{ such that } k_{i,s_i} \notin L_i; \\ f_{l_{1,t_1}, l_{2,t_2}, \dots, l_{n,t_n}}, & \text{if for each } i: p_{i,q_i} = l_{i,t_i} \in K_i \\ & \text{and there exists } i \text{ such that } l_{i,t_i} \notin K_i; \\ f_{l_{1,t_1}, l_{2,t_2}, \dots, l_{n,t_n}} (a_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}}), & \text{if for each } i: \\ & p_{i,q_i} = k_{i,s_i} = l_{i,t_i} \in K_i \cap L_i; \\ e_o, & \text{otherwise,} \end{cases}$$

where a - and f -elements belong to sets with equal dimensions and $e_o = 0$ or $e_o = \perp$.

Operation $\boxtimes$

$$A \boxtimes F = [K_1 \cap L_1, K_2 \cap L_2, \dots, K_n \cap L_n, \{h_{p_{1,q_1}, p_{2,q_2}, \dots, p_{n,q_n}}\}],$$

where

$$h_{p_{1,q_1}, p_{2,q_2}, \dots, p_{n,q_n}} = a_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}} \circ f_{l_{1,t_1}, l_{2,t_2}, \dots, l_{n,t_n}},$$

if for each $i: p_{i,q_i} = k_{i,s_i} = l_{i,t_i} \in K_i \cap L_i$, where $\circ \in \{+, -, \times, :\}$ and $a_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}} \in \mathbb{R}$.

$$F \boxtimes A = [K_1 \cap L_1, K_2 \cap L_2, \dots, K_n \cap L_n, \{h_{p_{1,q_1}, p_{2,q_2}, \dots, p_{n,q_n}}\}],$$

where

$$h_{p_{1,q_1}, p_{2,q_2}, \dots, p_{n,q_n}} = f_{l_{1,t_1}, l_{2,t_2}, \dots, l_{n,t_n}} (a_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}}),$$

if for each $i: p_{i,q_i} = k_{i,s_i} = l_{i,t_i} \in K_i \cap L_i$, and a - and f -elements belong to sets with equal dimensions.

Finally, for the two n -DEIMFEs F and G , following [21] we define two operations “composition” by:

Operation $\square_{(\oplus)}$

$$F \square_{(\oplus)} G = [K_1 \cup L_1, K_2 \cup L_2, \dots, K_n \cup L_n, \{h_{p_1, q_1, p_2, q_2, \dots, p_n, q_n}\}],$$

where

$$h_{p_1, q_1, p_2, q_2, \dots, p_n, q_n} = \begin{cases} f_{k_1, s_1, k_2, s_2, \dots, k_n, s_n}, & \text{if for each } i: p_{i, q_i} = k_{i, s_i} \in K_i \\ & \text{and there exists } i \text{ such that } k_{i, s_i} \notin L_i; \\ \\ g_{l_1, t_1, l_2, t_2, \dots, l_n, t_n}, & \text{if for each } i: p_{i, q_i} = l_{i, t_i} \in K_i \\ & \text{and there exists } i \text{ such that } l_{i, t_i} \notin K_i; \\ \\ f_{k_1, s_1, k_2, s_2, \dots, k_n, s_n}(g_{l_1, t_1, l_2, t_2, \dots, l_n, t_n}), & \text{if for each } i: p_{i, q_i} = k_{i, s_i} = l_{i, t_i} \\ & \in K_i \cap L_i; \\ \\ \perp, & \text{otherwise} \end{cases}$$

Operation $\square_{(\otimes)}$

$$F \square_{(\otimes)} G = [K_1 \cap L_1, K_2 \cap L_2, \dots, K_n \cap L_n, \{h_{p_1, q_1, p_2, q_2, \dots, p_n, q_n}\}],$$

where

$$h_{p_1, q_1, p_2, q_2, \dots, p_n, q_n} = f_{k_1, s_1, k_2, s_2, \dots, k_n, s_n}(g_{l_1, t_1, l_2, t_2, \dots, l_n, t_n}),$$

where for each i : $p_{i, q_i} = k_{i, s_i} = l_{i, t_i} \in K_i \cap L_i$ and $e_0 = 0$ or $e_0 = \perp$.

When the elements of the n -DEIMFE are propositions or predicates, the elements of set $\mathcal{F}^d$ will be logical functions.

3.7 Relations over n -DEIMs

Let the two n -DEIMs A and B be given. Following [37], we will introduce some definitions where $\subset$ and $\subseteq$ denote the relations “*strong inclusion*” and “*weak inclusion*”.

We will discuss two cases for the elements of the n -DEIMs.

First case: let the elements of the n -DEIMs are real or natural numbers, or elements of set $\{0, 1\}$.

The strict relation “inclusion about dimension” is

$$A \subset_d B \text{ iff } (\forall i)(K_i \subset L_i) \vee (\forall i)(K_i \subseteq L_i \ \& \ (\exists j : 1 \leq j \leq n)(K_j \neq L_j)) \\ \& (\forall i)(\forall j : 1 \leq j \leq m_i)(a_{k_1, s_1, k_2, s_2, \dots, k_n, s_n} = b_{k_1, s_1, k_2, s_2, \dots, k_n, s_n}).$$

The non-strict relation “inclusion about dimension” is

$$A \subseteq_d B \text{ iff } (\forall i)(K_i \subseteq L_i) \& (\forall i)(\forall j : 1 \leq j \leq m_i) \\ (a_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}} = b_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}}).$$

The strict relation “inclusion about value” is

$$A \subset_v B \text{ iff } (\forall i)(K_i = L_i) \& (\forall i)(\forall s_i : 1 \leq s_i \leq m_i) \\ (a_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}} < b_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}}).$$

The non-strict relation “inclusion about value” is

$$A \subseteq_v B \text{ iff } (\forall i)(K_i = L_i) \& (\forall i)(\forall s_i : 1 \leq s_i \leq m_i) \\ (a_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}} \leq b_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}}).$$

The strict relation “inclusion” is

$$A \subset B \text{ iff } (\forall i)(K_i \subset L_i) \vee (\forall i)(K_i \subseteq L_i \& (\exists j : 1 \leq j \leq n)(K_j \neq L_j)) \\ \& (\forall i)(\forall j : 1 \leq j \leq m_i)(a_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}} < b_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}}).$$

The non-strict relation “inclusion” is

$$A \subseteq B \text{ iff } (\forall i)(K_i \subseteq L_i) \& (\forall i)(\forall j : 1 \leq j \leq m_i) \\ (a_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}} \leq b_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}}).$$

Second case: let V be an evaluation function that estimates the truth-value of logical variables, propositions or predicates. Then, the first two above relations preserve their form, but the four next relations change their forms, as follows.

The strict relation “inclusion about value” is

$$A \subset_v B \text{ iff } (\forall i)(K_i = L_i) \& (\forall i)(\forall s_i : 1 \leq s_i \leq m_i) \\ (V(a_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}}) < V(b_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}})).$$

The non-strict relation “inclusion about value” is

$$A \subseteq_v B \text{ iff } (\forall i)(K_i = L_i) \& (\forall i)(\forall s_i : 1 \leq s_i \leq m_i) \\ (V(a_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}}) \leq V(b_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}})).$$

The strict relation “inclusion” is

$$A \subset B \text{ iff } (\forall i)(K_i \subset L_i) \vee (\forall i)(K_i \subseteq L_i \& (\exists j : 1 \leq j \leq n)(K_j \neq L_j)) \\ \& (\forall i)(\forall j : 1 \leq j \leq m_i)(V(a_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}}) < V(b_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}})).$$

The non-strict relation “inclusion” is

$$A \subseteq B \text{ iff } (\forall i)(K_i \subseteq L_i) \& (\forall i)(\forall j : 1 \leq j \leq m_i) \\ (V(a_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}}) \leq V(b_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}})).$$

3.8 n -Dimensional Hierarchical Index Matrices (n -DHIMs)

Now, following [37], we describe the case when A is an n -dimensional HIM with elements that are also HIMs.

Let

$$A = [K_1, K_2, \dots, K_n, \{a_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}}\}]$$

where $K_i = \{k_{i,1}, k_{i,2}, \dots, k_{i,m_i}\}$, $m_i \geq 1$ and each one of its elements has the form

$$a_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}} = [L_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}}^1, L_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}}^2, \dots, L_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}}^{r_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}}}, \{b_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}}^{q_1, q_2, \dots, q_{r_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}}}}\}],$$

where

- $\langle k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n} \rangle \in K_1 \times K_2 \times \dots \times K_n$,
- $r_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}} \geq 1$ is the dimension of the given HIM $a_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}}$,
- for each i ($1 \leq i \leq n$) and for each j ($1 \leq j \leq r_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}}$):

$$1 \leq q_j \leq |L_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}}^j|$$

for $|X|$ being the cardinality of set X and

$$K_i \cap L_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}}^j = \emptyset, \quad (3.1)$$

- for each j_1, j_2 ($1 \leq j_1 < j_2 \leq r_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}}$):

$$L_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}}^{j_1} \cap L_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}}^{j_2} = \emptyset. \quad (3.2)$$

Then

$$A|_* = \left[\bigcup_{\langle k_{1,s_1}, \dots, k_{n,s_n} \rangle \in K_1 \times \dots \times K_n} L_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}}^1, \dots, \bigcup_{\langle k_{1,s_1}, \dots, k_{n,s_n} \rangle \in K_1 \times \dots \times K_n} L_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}}^n, \{c_{u_{1,v_1}, \dots, u_{n,v_n}}^{w_1, \dots, w_n}\} \right],$$

where

$$c_{u_{1,v_1}, \dots, u_{n,v_n}}^{w_1, \dots, w_n} = \begin{cases} b_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}}^{q_1, q_2, \dots, q_{r_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}}}}, & \text{if } w_i = q_i, \text{ and } u_{i,v_i} = k_{i,s_i} \\ & \text{for } 1 \leq i \leq n, \\ \perp, & \text{otherwise.} \end{cases}.$$

We can formulate and prove the following assertion. Let

$$A = [K_1, K_2, \dots, K_n, \{a_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}}\}]$$

be an n -dimensional HIM, let each of its elements $a_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}}$ be an n -dimensional HIM and let the index sets of all a -HIMs satisfy (3.1) and (3.2). Then

$$A|^* = \sum_{\langle k_{1,s_1}, \dots, k_{n,s_n} \rangle \in K_1 \times \dots \times K_n} a_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}}.$$

We can change condition (3.2) with the weaker one: for every two different n -tuples $\langle k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n} \rangle$ and $\langle l_{1,t_1}, l_{2,t_2}, \dots, l_{n,t_n} \rangle$:

$$\bigcap_{i=1}^n L_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}}^i \cap \bigcap_{i=1}^n L_{l_{1,t_1}, l_{2,t_2}, \dots, l_{n,t_n}}^i = \emptyset.$$

3.9 Cartesian product over n -DEIMs

Initially, following [44], we will discuss case for Cartesian product over two 2-dimensional EIMs.

Let us have two EIMs $A = [K, L, \{a_{k_i, l_j}\}]$ and $B = [P, Q, \{b_{p_r, q_s}\}]$.

The first type of Cartesian product is the following:

$$A \times_C B = [K \times P, L \times Q, \{c_{\langle k_i, p_r \rangle, \langle l_j, q_s \rangle}\}],$$

where

$$c_{\langle \langle k_i, p_r \rangle, \langle l_j, q_s \rangle \rangle} = \langle a_{k_i, l_j}, b_{p_r, q_s} \rangle$$

and operation $\times$ between K and P , and between L and Q is the standard set-theoretical Cartesian product.

The second type of Cartesian product is the following:

$$A \times_{(\circ)} B = [K \times P, L \times Q, \{c_{\langle k_i, p_r \rangle, \langle l_j, q_s \rangle}\}],$$

where

$$c_{\langle \langle k_i, p_r \rangle, \langle l_j, q_s \rangle \rangle} = a_{k_i, l_j} \circ b_{p_r, q_s},$$

where $\circ$ is a suitable operation.

Now, following [36], we will discuss the general case.

Let us have two n -DEIMs

$$\begin{aligned} A &= [K_1, K_2, \dots, K_n, \{a_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}}\}], \\ B &= [L_1, L_2, \dots, L_n, \{b_{l_{1,t_1}, l_{2,t_2}, \dots, l_{n,t_n}}\}], \end{aligned}$$

and let $X \times Y$ be the standard set-theoretical Cartesian product.

First, we introduce the Cartesian product

$$A \times_C B = [K_1 \times L_1, K_2 \times L_2, \dots, K_n \times L_n, \{\langle a_{k_1, s_1}, k_{2, s_2}, \dots, k_{n, s_n}, b_{l_1, t_1}, l_{2, t_2}, \dots, l_{n, t_n} \rangle\}].$$

When $n = 2$, we obtain the first Cartesian product from for the case of two IFIMs.

Let for the three elements $a, b, c \in \mathcal{X}$ the equalities

$$\langle \langle a, b \rangle, c \rangle = \langle a, b, c \rangle = \langle a, \langle b, c \rangle \rangle$$

be valid.

The operation $\times_C$ can be modified for two EIMs so that the first one is an m -DEIM and the second one is an n -DEIM. Let

$$A = [K_1, K_2, \dots, K_m, \{a_{k_1, s_1}, k_{2, s_2}, \dots, k_{m, s_m}\}]$$

and B has the above form. Then

$$A \times^C B = [K_1, K_2, \dots, K_m, L_1, L_2, \dots, L_n, \{\langle a_{k_1, s_1}, k_{2, s_2}, \dots, k_{m, s_m}, b_{l_1, t_1}, l_{2, t_2}, \dots, l_{n, t_n} \rangle\}].$$

The following two Cartesian products are extensions of the two above operations:

$$\begin{aligned} A \times_{C, \circ} B &= [K_1 \times L_1, K_2 \times L_2, \dots, K_n \times L_n, \{a_{k_1, s_1}, k_{2, s_2}, \dots, k_{n, s_n} \circ b_{l_1, t_1}, l_{2, t_2}, \dots, l_{n, t_n}\}]; \\ A \times_{\circ}^C B &= [K_1, K_2, \dots, K_m, L_1, L_2, \dots, L_n, \{a_{k_1, s_1}, k_{2, s_2}, \dots, k_{m, s_m} \circ b_{l_1, t_1}, l_{2, t_2}, \dots, l_{n, t_n}\}]. \end{aligned}$$

In [36] it is proved that these four operations are also associative, but not commutative.

Let for every three objects $a, b, c \in \mathcal{X}$:

$$a \circ (b * c) = (a \circ b) * (a \circ c)$$

and/or

$$(a * b) \circ c = (a \circ c) * (b \circ c)$$

In [36] it is proved that for every three EIM A, B, C :

$$A \times_{\circ}^C (B \times_{C, *} C) = (A \times_{\circ}^C B) \times_{C, *} (A \times_{\circ}^C C)$$

and

$$(A \times_{C, *} B) \times_{\circ}^C C = (A \times_{\circ}^C C) \times_{C, *} (B \times_{\circ}^C C),$$

and it is also proved that

$$A \times_{C, \circ} (B \times_{C, *} C) = (A \times_{C, \circ} B) \times_{C, *} (A \times_{C, \circ} C)$$

and

$$(A \times_{C, *} B) \times_{C, \circ} C = (A \times_{C, \circ} C) \times_{C, *} (B \times_{C, \circ} C).$$

Now, we introduce an object that can be interpreted as the result of the operation Cartesian product.

Let the following m in number multi-DEIMs be given for $i = 1, 2, \dots, m$, i.e., the different EIMs can have different dimensions, i.e., $n_1, n_2, \dots, n_m$:

$$A_i = [K; L_1^i, L_2^i, \dots, L_{s_i}^i \{a_{k, l_1^i, \dots, l_{r_i}^i}\}]$$

so that the index set K is equal for all of them, while the sets $L_1^1, L_2^1, \dots, L_{s_m}^m$ do not have equal elements. Now we define

$$\Delta(A_1, A_2, \dots, A_m) = [K; L_1^1, \dots, L_{s_m}^m, \{b_{k, l_1^1, l_2^1, \dots, l_{r_m}^m}\}],$$

where for each $i : 1 \leq i \leq m$

$$b_{k, l_1^1, l_2^1, \dots, l_{r_i}^i, \dots, l_{r_m}^m} = a_{k, l_1^i, \dots, l_{r_i}^i}.$$

Geometrically, this object can be represented as shown on Figure 3.1. By this reason we can call it an IM-Book (IMB). This object can be extended in two directions.

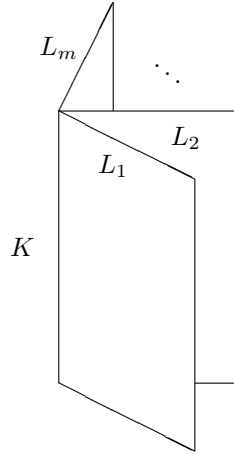


Figure 3.1. Geometrical interpretation of $\Delta(A_1, A_2, \dots, A_m)$

First direction is obtained when we check the index set K of the EIMs $A_1, A_2, \dots, A_m$ with a set (equal for each EIMs) $K_1, K_2, \dots, K_k$, while the second direction is to omit the condition that sets $L_1^1, L_2^1, \dots, L_{s_m}^m$ do not have equal elements, changing it with the possibility the sets L_p^i and L_q^i to have joint elements for $p \neq q$, but this will be not valid for the sets L_r^p and L_r^q for $p \neq q$. In this case, the result of applying of operator Δ over EIMs $A_1, A_2, \dots, A_m$ has two forms:

$$\Delta_{\cup}(A_1, A_2, \dots, A_m) = [K; \bigcup_{i=1}^m L_1^i, \bigcup_{i=1}^m L_2^i, \dots, \bigcup_{i=1}^m L_{s_i}^i \{b_{k, l_1^1, l_2^1, \dots, l_{r_m}^m}\}]$$

and

$$\Delta_{\cap}(A_1, A_2, \dots, A_m) = [K; \bigcap_{i=1}^m L_1^i, \bigcap_{i=1}^m L_2^i, \dots, \bigcap_{i=1}^m L_{s_i}^i \{b_{k, l_1^1, l_2^1, \dots, l_{r_m}^m}\}],$$

where some of the sets L_j^i can be the empty set.

3.10 Example from graph theory. Part 2

Let us have the following multi-graph

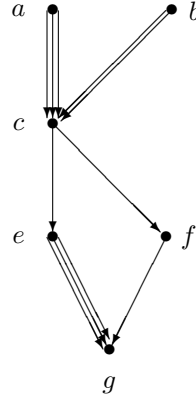


Figure 3.2. Multigraph

Now, the adjacency IM has the form

$$P = \begin{array}{c|cccc} & c & e & f & g \\ \hline a & 3 & 0 & 0 & 0 \\ b & 2 & 0 & 0 & 0 \\ c & 0 & 1 & 1 & 0 \\ e & 0 & 0 & 0 & 3 \\ f & 0 & 0 & 0 & 1 \end{array}.$$

In a similar way we can represent a weighted graph. When we apply to AdIM P operations “Sum-row-aggregation” and “Sum-column-aggregation”, we obtain the non-(0,1)-IMs that show how many arcs enter and how many arcs leave the individual vertices, respectively:

$$\sigma_{\text{sum}}(P, k) = \frac{\begin{array}{c|cccc} & c & e & f & g \\ \hline k & 5 & 1 & 1 & 4 \end{array}}{\quad} \quad \text{and} \quad \rho_{\text{sum}}(P, l) = \frac{\begin{array}{c|c} & l \\ \hline a & 3 \\ b & 2 \\ c & 2 \\ e & 3 \\ f & 1 \end{array}}{\quad}$$

Now, let as in Section 2.7, the arcs between vertices a and c coloured by yellow, green and red colours, these between vertices b and c – by green and blue, between vertices c and e – by green, between vertices c and f – by blue, between vertices e and g are coloured by black, green and red colours, and between vertices f and g – by green. Then we can construct a 3DIM with index sets: $\{a, b, c, e, f\}$, $\{c, e, f, g\}$ and $\{\text{white}, \text{green}, \text{red}, \text{blue}, \text{black}\}$. Therefore, it will have dimension $5 \times 4 \times 5$ and it can be represented by the following five 2-DIMs:

<i>yellow</i>	<i>c</i>	<i>e</i>	<i>f</i>	<i>g</i>	<i>green</i>	<i>c</i>	<i>e</i>	<i>f</i>	<i>g</i>	<i>red</i>	<i>c</i>	<i>e</i>	<i>f</i>	<i>g</i>
<i>a</i>	1	⊥	⊥	⊥	<i>a</i>	1	⊥	⊥	⊥	<i>a</i>	1	⊥	⊥	⊥
<i>b</i>	⊥	⊥	⊥	⊥	<i>b</i>	1	⊥	⊥	⊥	<i>b</i>	⊥	⊥	⊥	⊥
<i>c</i>	⊥	⊥	⊥	⊥	<i>c</i>	⊥	1	⊥	⊥	<i>c</i>	⊥	⊥	⊥	⊥
<i>e</i>	⊥	⊥	⊥	⊥	<i>e</i>	⊥	⊥	⊥	1	<i>e</i>	⊥	⊥	⊥	1
<i>f</i>	⊥	⊥	⊥	⊥	<i>f</i>	⊥	⊥	⊥	1	<i>f</i>	⊥	⊥	⊥	⊥

<i>blue</i>	<i>c</i>	<i>e</i>	<i>f</i>	<i>g</i>	<i>black</i>	<i>c</i>	<i>e</i>	<i>f</i>	<i>g</i>
<i>a</i>	⊥	⊥	⊥	⊥	<i>a</i>	⊥	⊥	⊥	⊥
<i>b</i>	1	⊥	⊥	⊥	<i>b</i>	⊥	⊥	⊥	⊥
<i>c</i>	⊥	⊥	1	⊥	<i>c</i>	⊥	⊥	⊥	⊥
<i>e</i>	⊥	⊥	⊥	⊥	<i>e</i>	⊥	⊥	⊥	1
<i>f</i>	⊥	⊥	⊥	⊥	<i>f</i>	⊥	⊥	⊥	⊥

But we can also construct another 3-DIM with dimensions $5 \times 4 \times 3$, where the third dimension is indexed with numbers of the colour orders $\{1, 2, 3\}$ and it can be represented by the following three 2-DIMs (the distribution of colours per 2-DIMs follows the order of listing in the text, respectively):

1	<i>c</i>	<i>e</i>	<i>f</i>	<i>g</i>	2	<i>c</i>	<i>e</i>	<i>f</i>	<i>g</i>	3	<i>c</i>	<i>e</i>	<i>f</i>	<i>g</i>
<i>a</i>	<i>yellow</i>	⊥	⊥	⊥	<i>a</i>	<i>green</i>	⊥	⊥	⊥	<i>a</i>	<i>red</i>	⊥	⊥	⊥
<i>b</i>	<i>green</i>	⊥	⊥	⊥	<i>b</i>	<i>blue</i>	⊥	⊥	⊥	<i>b</i>	⊥	⊥	⊥	⊥
<i>c</i>	⊥	<i>green</i>	<i>blue</i>	⊥	<i>c</i>	⊥	⊥	⊥	⊥	<i>c</i>	⊥	⊥	⊥	⊥
<i>e</i>	⊥	⊥	⊥	<i>black</i>	<i>e</i>	⊥	⊥	⊥	<i>green</i>	<i>e</i>	⊥	⊥	⊥	<i>red</i>
<i>f</i>	⊥	⊥	⊥	<i>green</i>	<i>f</i>	⊥	⊥	⊥	⊥	<i>f</i>	⊥	⊥	⊥	⊥

We finish this Section with the following

Open problem: To represent the basic concepts related to graphs (e.g., path, diameter, etc.) and properties (e.g., planarity, (dis)connectness, symmetry, etc.) of the graphs in IM-form.

Intuitionistic Fuzzy Index Matrices (IFIMs)

In this chapter, following [11,15,20], we extend the concept of IM, introducing the concept of an Intuitionistic Fuzzy IM (IFIM) and an Extended IFIM (EIFIM)

Initially, we give some remarks on Intuitionistic Fuzzy Sets (IFSs, see, e.g., [7,13]), Intuitionistic Fuzzy Logics (IFLs, see, e.g., [26]) and especially, of their particular case, Intuitionistic Fuzzy Pairs (IFPs; see [26,50]).

4.1 Remarks on intuitionistic fuzziness

The IFP is an object with the form $\langle a, b \rangle$, where $a, b \in [0, 1]$ and $a + b \leq 1$, that is used as an evaluation of some object or process. Its components (a and b) are interpreted as degrees of membership and non-membership, or degrees of validity and non-validity, or degree of correctness and non-correctness, etc.

When we have two IFPs $x = \langle a, b \rangle$ and $y = \langle c, d \rangle$, multiple operations, relations and operators are defined over them in [26,50]. Here, we will use only the following:

Relations:

$$\begin{aligned} x < y & \text{ iff } a < c \text{ and } b > d \\ x \leq y & \text{ iff } a \leq c \text{ and } b \geq d \\ x = y & \text{ iff } a = c \text{ and } b = d \\ x \geq y & \text{ iff } a \geq c \text{ and } b \leq d \\ x > y & \text{ iff } a > c \text{ and } b < d; \end{aligned}$$

Operations:

$$\begin{aligned} \neg x &= \langle b, a \rangle \\ x \wedge y &= \langle \min(a, c), \max(b, d) \rangle \\ x \vee y &= \langle \max(a, c), \min(b, d) \rangle \\ x + y &= \langle a + c - a.c, b.d \rangle \\ x.y &= \langle a.c, b + d - b.d \rangle \end{aligned}$$

If, for example, $\circ$ is one of the above operations, then it can be interpreted as an ordered pair $\langle \circ_1, \circ_2 \rangle$ for suitable sub-operations $\circ_1$ and $\circ_2$, and

$$\langle a, b \rangle \circ \langle c, d \rangle = \langle a \circ_1 c, b \circ_2 d \rangle.$$

For example, if $\circ$ is the operation $\vee$, then its sub-operations will be $\langle \max, \min \rangle$, while if $\circ$ is the operation $\wedge$, then its sub-operations will be $\langle \min, \max \rangle$.

Operators:

$$\Box x = \langle a, 1 - a \rangle$$

$$\Diamond x = \langle 1 - b, b \rangle$$

$$D_\alpha x = \langle a + \alpha(1 - a - b), b + (1 - \alpha)(1 - a - b) \rangle, \text{ for } \alpha \in [0, 1]$$

$$F_{\alpha, \beta} x = \langle a + \alpha(1 - a - b), b + \beta(1 - a - b) \rangle, \text{ for } \alpha, \beta, \alpha + \beta \in [0, 1]$$

$$G_{\alpha, \beta} x = \langle \alpha a, \beta b \rangle, \text{ for } \alpha, \beta \in [0, 1]$$

$$H_{\alpha, \beta} x = \langle \alpha a, b + \beta(1 - a - b) \rangle, \text{ for } \alpha, \beta \in [0, 1]$$

$$H_{\alpha, \beta}^* x = \langle \alpha a, b + \beta(1 - \alpha a - b) \rangle, \text{ for } \alpha, \beta \in [0, 1]$$

$$\overline{H}_{\alpha, \beta} x = \langle \alpha a, b + b - \beta b \rangle, \text{ for } \alpha, \beta \in [0, 1]$$

$$J_{\alpha, \beta} x = \langle a + \alpha(1 - a - b), \beta b \rangle, \text{ for } \alpha, \beta \in [0, 1]$$

$$J_{\alpha, \beta}^* x = \langle a + \alpha(1 - a - \beta b), \beta b \rangle, \text{ for } \alpha, \beta \in [0, 1]$$

$$\overline{J}_{\alpha, \beta} x = \langle a + \alpha - \alpha a, \beta b \rangle, \text{ for } \alpha, \beta \in [0, 1]$$

$$P_{\alpha, \beta} x = \langle \max(\alpha, a), \min(\beta, b) \rangle, \text{ for } \alpha, \beta, \alpha + \beta \in [0, 1]$$

$$Q_{\alpha, \beta} x = \langle \min(\alpha, a), \max(\beta, b) \rangle, \text{ for } \alpha, \beta, \alpha + \beta \in [0, 1]$$

Currently there are more than 200 different operations “intuitionistic fuzzy implication” and more than 50 different operations “intuitionistic fuzzy negation”. In some of these definitions, we use the functions sg and $\overline{\text{sg}}$ that are defined by:

$$\text{sg}(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x \leq 0 \end{cases}$$

and

$$\overline{\text{sg}}(x) = \begin{cases} 0, & \text{if } x > 0 \\ 1, & \text{if } x \leq 0 \end{cases}$$

We will give only the first two definitions of negations:

$$\begin{aligned}\neg_1\langle a, b \rangle &= \langle b, a \rangle, \\ \neg_2\langle a, b \rangle &= \langle x, \overline{\text{sg}}(a), \text{sg}(a) \rangle\end{aligned}$$

For these negations, it is interesting that the first one satisfies the De Morgan law in the form

$$\neg_1(\neg_1 x \wedge \neg_1 y) = x \vee y \quad \text{and} \quad \neg_1(\neg_1 x \vee \neg_1 y) = x \wedge y$$

while the second one does not satisfy this law in the above form, but in the form

$$\neg_2(\neg_2 x \& \neg_2 y) = \neg_2 \neg_2 x \vee \neg_2 \neg_2 y \quad \text{and} \quad \neg_2(\neg_2 x \vee \neg_2 y) = \neg_2 \neg_2 x \wedge \neg_2 \neg_2 y,$$

because $\neg_2 \neg_2 x \neq x$, while $\neg_1 \neg_1 x = x$.

Let a set E be fixed. An Intuitionistic Fuzzy Set (IFS) A in E is an object of the following form (see, e.g., [7, 13]):

$$A = \{\langle x, \mu_A(x), \nu_A(x) \rangle \mid x \in E\},$$

where the functions $\mu_A : E \rightarrow [0, 1]$ and $\nu_A : E \rightarrow [0, 1]$ define the degree of membership and the degree of non-membership of the element $x \in E$, respectively, and for every $x \in E$:

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1.$$

The operators over IFPs now have the following forms for $\alpha, \beta \in [0, 1]$:

$$\begin{aligned}\Box A &= \{\langle x, \mu_A(x), 1 - \mu_A(x) \rangle \mid x \in E\}; \\ \Diamond A &= \{\langle x, 1 - \nu_A(x), \nu_A(x) \rangle \mid x \in E\}; \\ D_\alpha(A) &= \{\langle x, \mu_A(x) + \alpha\pi_A(x), \nu_A(x) + (1 - \alpha)\pi_A(x) \rangle \mid x \in E\}; \\ F_{\alpha, \beta}(A) &= \{\langle x, \mu_A(x) + \alpha\pi_A(x), \nu_A(x) + \beta\pi_A(x) \rangle \mid x \in E\}, \text{ where } \alpha + \beta \leq 1; \\ G_{\alpha, \beta}(A) &= \{\langle x, \alpha\mu_A(x), \beta\nu_A(x) \rangle \mid x \in E\}; \\ H_{\alpha, \beta}(A) &= \{\langle x, \alpha\mu_A(x), \nu_A(x) + \beta\pi_A(x) \rangle \mid x \in E\}, \\ H_{\alpha, \beta}^*(A) &= \{\langle x, \alpha\mu_A(x), \nu_A(x) + \beta(1 - \alpha\mu_A(x) - \nu_A(x)) \rangle \mid x \in E\}; \\ \overline{H}_{\alpha, \beta}(A) &= \{\langle x, \alpha\mu_A(x), \nu_A(x) + \beta - \beta\nu_A(x) \rangle \mid x \in E\}, \\ J_{\alpha, \beta}(A) &= \{\langle x, \mu_A(x) + \alpha\pi_A(x), \beta\nu_A(x) \rangle \mid x \in E\}; \\ J_{\alpha, \beta}^*(A) &= \{\langle x, \mu_A(x) + \alpha(1 - \mu_A(x) - \beta\nu_A(x)), \beta\nu_A(x) \rangle \mid x \in E\}; \\ \overline{J}_{\alpha, \beta}(A) &= \{\langle x, \mu_A(x) + \alpha - \alpha\nu_A(x), \beta\nu_A(x) \rangle \mid x \in E\}; \\ P_{\alpha, \beta}(A) &= \{\langle x, \max(\mu_A(x), \alpha), \min(\nu_A(x), \beta) \rangle \mid x \in E\}, \text{ where } \alpha + \beta \leq 1; \\ Q_{\alpha, \beta}(A) &= \{\langle x, \min(\mu_A(x), \alpha), \max(\nu_A(x), \beta) \rangle \mid x \in E\}, \text{ where } \alpha + \beta \leq 1.\end{aligned}$$

In [13], the following operator is further described:

$$N_{\alpha,\beta}(A) = \{ \langle x, \mu_A(x), \nu_A(x) \rangle \mid \mu_A(x) \geq \alpha \ \& \ \nu_A(x) \leq \beta \ \& \ x \in E \},$$

where $\alpha + \beta \leq 1$.

4.2 Definition of an IFIM and operations over IFIMs

When $\mathcal{X}$ is a set of IFPs, then the IM with elements from $\mathcal{X}$ is called an Intuitionistic fuzzy IM (IFIM).

Let I be a fixed set. By IFIM with index sets K and L ($K, L \subset I$), we denote the object:

$$A = [K, L, \{ \langle \mu_{k_i, l_j}, \nu_{k_i, l_j} \rangle \}]$$

$$= \begin{array}{c|cccc} & l_1 & \dots & l_j & \dots & l_n \\ \hline k_1 & \langle \mu_{k_1, l_1}, \nu_{k_1, l_1} \rangle & \dots & \langle \mu_{k_1, l_j}, \nu_{k_1, l_j} \rangle & \dots & \langle \mu_{k_1, l_n}, \nu_{k_1, l_n} \rangle \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_i & \langle \mu_{k_i, l_1}, \nu_{k_i, l_1} \rangle & \dots & \langle \mu_{k_i, l_j}, \nu_{k_i, l_j} \rangle & \dots & \langle \mu_{k_i, l_n}, \nu_{k_i, l_n} \rangle \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_m & \langle \mu_{k_m, l_1}, \nu_{k_m, l_1} \rangle & \dots & \langle \mu_{k_m, l_j}, \nu_{k_m, l_j} \rangle & \dots & \langle \mu_{k_m, l_n}, \nu_{k_m, l_n} \rangle \end{array},$$

where for every $1 \leq i \leq m, 1 \leq j \leq n$: $0 \leq \mu_{k_i, l_j}, \nu_{k_i, l_j}, \mu_{k_i, l_j} + \nu_{k_i, l_j} \leq 1$ and

$$K = \{k_1, k_2, \dots, k_m\},$$

$$L = \{l_1, l_2, \dots, l_n\}.$$

Let the IFIMs A and $B = [P, Q, \{ \langle \rho_{p_r, q_s}, \sigma_{p_r, q_s} \rangle \}]$ be given. Over them we can define the following operations:

(max,min)-Addition or ($\vee$)-Addition

$$A \oplus_{(\vee)} B = A \oplus_{(\max, \min)} B = [K \cup P, L \cup Q, \{ \langle \varphi_{t_u, v_w}, \psi_{t_u, v_w} \rangle \}],$$

where

$$\langle \varphi_{t_u, v_w}, \psi_{t_u, v_w} \rangle = \langle \mu_{k_i, l_j} \vee \rho_{p_r, q_s}, \nu_{k_i, l_j} \wedge \sigma_{p_r, q_s} \rangle$$

$$= \begin{cases} \langle \mu_{k_i, l_j}, \nu_{k_i, l_j} \rangle, & \text{if } t_u = k_i \in K \text{ and } v_w = l_j \in L - Q \\ & \text{or } t_u = k_i \in K - P \text{ and } v_w = l_j \in L; \\ \langle \rho_{p_r, q_s}, \sigma_{p_r, q_s} \rangle, & \text{if } t_u = p_r \in P \text{ and } v_w = q_s \in Q - L \\ & \text{or } t_u = p_r \in P - K \text{ and } v_w = q_s \in Q; \\ \langle \max(\mu_{k_i, l_j}, \rho_{p_r, q_s}), \min(\nu_{k_i, l_j}, \sigma_{p_r, q_s}) \rangle, & \text{if } t_u = k_i = p_r \in K \cap P \\ & \text{and } v_w = l_j = q_s \in L \cap Q; \\ \langle 0, 1 \rangle, & \text{otherwise.} \end{cases}$$

Addition-(min,max) or Addition-($\wedge$)

$$A \oplus_{(\min, \max)} B = [K \cup P, L \cup Q, \{\langle \varphi_{t_u, v_w}, \psi_{t_u, v_w} \rangle\}],$$

where

$$= \begin{cases} \langle f_{t_u, v_w}, \psi_{t_u, v_w} \rangle = \langle \mu_{k_i, l_j} \wedge \rho_{p_r, q_s}, \nu_{k_i, l_j} \vee \sigma_{p_r, q_s} \rangle \\ \mu_{k_i, l_j}, \nu_{k_i, l_j}, & \text{if } t_u = k_i \in K \text{ and } v_w = l_j \in L - Q \\ & \text{or } t_u = k_i \in K - P \text{ and } v_w = l_j \in L; \\ \langle \rho_{p_r, q_s}, \sigma_{p_r, q_s} \rangle, & \text{if } t_u = p_r \in P \text{ and } v_w = q_s \in Q - L \\ & \text{or } t_u = p_r \in P - K \text{ and } v_w = q_s \in Q; \\ \langle \min(\mu_{k_i, l_j}, \rho_{p_r, q_s}), \max(\nu_{k_i, l_j}, \sigma_{p_r, q_s}) \rangle, & \text{if } t_u = k_i = p_r \in K \cap P \\ & \text{and } v_w = l_j = q_s \in L \cap Q; \\ \langle 0, 1 \rangle, & \text{otherwise.} \end{cases}$$

Termwise multiplication-(max,min) or Termwise multiplication-($\vee$)

$$A \otimes_{(\vee)} B = A \otimes_{(\max, \min)} B = [K \cap P, L \cap Q, \{\langle \varphi_{t_u, v_w}, \psi_{t_u, v_w} \rangle\}],$$

where

$$\begin{aligned} \langle \varphi_{t_u, v_w}, \psi_{t_u, v_w} \rangle &= \langle \mu_{k_i, l_j} \vee \rho_{p_r, q_s}, \nu_{k_i, l_j} \wedge \sigma_{p_r, q_s} \rangle \\ &= \langle \max(\mu_{k_i, l_j}, \rho_{p_r, q_s}), \min(\nu_{k_i, l_j}, \sigma_{p_r, q_s}) \rangle. \end{aligned}$$

Termwise multiplication-(min,max) or Termwise multiplication-($\wedge$)

$$A \otimes_{(\wedge)} B = A \otimes_{(\min, \max)} B = [K \cap P, L \cap Q, \{\langle \varphi_{t_u, v_w}, \psi_{t_u, v_w} \rangle\}],$$

where

$$\begin{aligned} \langle \varphi_{t_u, v_w}, \psi_{t_u, v_w} \rangle &= \langle \mu_{k_i, l_j} \wedge \rho_{p_r, q_s}, \nu_{k_i, l_j} \vee \sigma_{p_r, q_s} \rangle \\ &= \langle \min(\mu_{k_i, l_j}, \rho_{p_r, q_s}), \max(\nu_{k_i, l_j}, \sigma_{p_r, q_s}) \rangle. \end{aligned}$$

Classical negation of an IFIM

$$\neg A = [K, L, \{\neg \langle \mu_{k_i, l_j}, \nu_{k_i, l_j} \rangle\}] = [K, L, \{\langle \nu_{k_i, l_j}, \mu_{k_i, l_j} \rangle\}],$$

In [20], a lot of other operations are defined over an extension of the IFIMs, that in the particular case are valid for the IFIM, e.g., multiplication, subtraction, etc. Some relations are defined over IMs that do not exist for standard matrices. Below, we will give only the following two of them that are different from these in Section 1.8.

The strict relation “inclusion about value” is

$$\begin{aligned} A \subseteq_v B \text{ iff } (K^* = P^*) \ \& \ (L^* = Q^*) \ \& \ (\forall k \in K)(\forall l \in L) \\ & (\langle a_{k, l}, b_{k, l} \rangle < \langle c_{k, l}, d_{k, l} \rangle). \end{aligned}$$

The non-strict relation “inclusion about value” is

$$\begin{aligned} A \subseteq_v B \text{ iff } (K^* = P^*) \ \& \ (L^* = Q^*) \ \& \ (\forall k \in K)(\forall l \in L) \\ & (\langle a_{k, l}, b_{k, l} \rangle \leq \langle c_{k, l}, d_{k, l} \rangle). \end{aligned}$$

4.3 Hierarchical IFIMs (HIFIMs)

Let $\circ$ be fixed operation (i.e., pair of operations and let A be a hierarchical IFIM with s levels). A has the form:

$$A = \langle A^1, \mu, \nu \rangle = [K^1, L^1, \{\langle A_{k_{i_1}, l_{j_1}}^2, \mu_{k_{i_1}, l_{j_1}}^1, \nu_{k_{i_1}, l_{j_1}}^1 \rangle\}]$$

$$= \begin{array}{c|ccc} & l_1^1 & \cdots & l_{j_1}^1 & \cdots & l_{n_1}^1 \\ \hline k_1^1 & & \cdots & & & \cdots \\ \vdots & \vdots & \ddots & \vdots & & \ddots \\ k_{i_1}^1 & & \langle A_{k_{i_1}, l_{j_1}}^2, \mu_{k_{i_1}, l_{j_1}}^1, \nu_{k_{i_1}, l_{j_1}}^1 \rangle & & & \\ \vdots & \vdots & \ddots & \vdots & & \ddots \\ k_{m_1}^1 & & \cdots & & & \cdots \end{array},$$

where

$$\begin{aligned} K^1 &= \{k_{i_1}^1, \dots, k_{i_1}^1, \dots, k_{m_1}^1\}, \\ L^1 &= \{l_{j_1}^1, \dots, l_{j_1}^1, \dots, l_{n_1}^1\}, \\ \langle \mu, \nu \rangle &=_{k_{i_1}, l_{j_1}^1} \langle \mu_{k_{i_1}, l_{j_1}^1}^1, \nu_{k_{i_1}, l_{j_1}^1}^1 \rangle, \end{aligned}$$

and for r -th level ($2 \leq r \leq s$) the IFIM will have the form:

$$\begin{aligned} & A_{k_{i_1}^1, l_{j_1}^1; k_{i_2}^2, l_{j_2}^2; \dots; k_{i_{r-1}}^{r-1}, l_{j_{r-1}}^{r-1}}^r \\ &= [K_{k_{i_1}^1, k_{i_2}^2, \dots, k_{i_{r-1}}^{r-1}}^r, L_{l_{j_1}^1, l_{j_2}^2, \dots, l_{j_{r-1}}^{r-1}}^r, \{\langle A_{k_{i_1}^1, l_{j_1}^1, k_{i_2}^2, l_{j_2}^2, \dots, k_{i_r}^r, l_{j_r}^r}^{r+1}, \\ & \quad \mu_{k_{i_1}^1, l_{j_1}^1, k_{i_2}^2, l_{j_2}^2, \dots, k_{i_r}^r, l_{j_r}^r}^r, \nu_{k_{i_1}^1, l_{j_1}^1, k_{i_2}^2, l_{j_2}^2, \dots, k_{i_r}^r, l_{j_r}^r}^r \rangle\}] \\ &= \begin{array}{c|ccc} & \cdots & \langle l_{j_1}^1, l_{j_2}^2, \dots, l_{j_r}^r \rangle & \cdots \\ \hline \langle k_{i_1}^1, k_{i_2}^2, \dots, k_{i_r}^r \rangle & \cdots & \cdots & \cdots \\ \vdots & \ddots & \vdots & \ddots \\ & \langle A_{k_{i_1}^1, l_{j_1}^1, k_{i_2}^2, l_{j_2}^2, \dots, k_{i_r}^r, l_{j_r}^r}^{r+1}, \\ & \mu_{k_{i_1}^1, l_{j_1}^1, k_{i_2}^2, l_{j_2}^2, \dots, k_{i_r}^r, l_{j_r}^r}^r, \\ & \nu_{k_{i_1}^1, l_{j_1}^1, k_{i_2}^2, l_{j_2}^2, \dots, k_{i_r}^r, l_{j_r}^r}^r \rangle & \\ \vdots & \ddots & \vdots & \ddots \\ \langle k_{i_1}^1, k_{i_2}^2, \dots, k_{i_r}^r \rangle & \cdots & \cdots & \cdots \end{array}, \end{aligned}$$

where for each $1 \leq u \leq r$: $1 \leq i_u \leq m_u$ and for each $1 \leq v \leq r$: $1 \leq j_v \leq n_v$ and

$$\begin{aligned}
& \langle \mu_{k_{i_1}^1, l_{j_1}^1; k_{i_2}^2, l_{j_2}^2; \dots; k_{i_{r-1}}^{r-1}, l_{j_{r-1}}^{r-1}} \nu_{k_{i_1}^1, l_{j_1}^1; k_{i_2}^2, l_{j_2}^2; \dots; k_{i_{r-1}}^{r-1}, l_{j_{r-1}}^{r-1}} \rangle \\
&= \overset{\circ}{k_{i_1}^1, l_{j_1}^1; k_{i_2}^2, l_{j_2}^2; \dots; k_{i_r}^r, l_{j_r}^r} \langle \mu_{k_{i_1}^1, l_{j_1}^1; k_{i_2}^2, l_{j_2}^2; \dots; k_{i_r}^r, l_{j_r}^r} \nu_{k_{i_1}^1, l_{j_1}^1; k_{i_2}^2, l_{j_2}^2; \dots; k_{i_r}^r, l_{j_r}^r} \rangle.
\end{aligned}$$

It is possible for some IFIM to be empty, i.e., to have the form

$$\frac{|\emptyset}{\emptyset \perp},$$

where the symbol “ $\perp$ ” means the absence of an element at that place.

Therefore, we can define an operator that juxtaposes to the HIFIM with $s+1$ levels an HIFIM with s levels. Let us have all HIFIMs with $(s+1)$ levels and let the $(k_{i_1}^1, l_{j_1}^1, k_{i_2}^2, l_{j_2}^2, \dots, k_{i_r}^r, l_{j_r}^r)$ -th from $(r+1)$ -st level $(1 \leq r \leq s)$ be

$$\langle A_{k_{i_1}^1, l_{j_1}^1; k_{i_2}^2, l_{j_2}^2; \dots; k_{i_r}^r, l_{j_r}^r}^{r+1}, \mu_{k_{i_1}^1, l_{j_1}^1; k_{i_2}^2, l_{j_2}^2; \dots; k_{i_r}^r, l_{j_r}^r}^r, \nu_{k_{i_1}^1, l_{j_1}^1; k_{i_2}^2, l_{j_2}^2; \dots; k_{i_r}^r, l_{j_r}^r}^r \rangle.$$

Then

$$\begin{aligned}
A_{k_{i_1}^1, l_{j_1}^1; k_{i_2}^2, l_{j_2}^2; \dots; k_{i_{r-1}}^{r-1}, l_{j_{r-1}}^{r-1}}^r &= [\overline{K}^r, \overline{L}^r, \{ \langle A_{k_{i_1}^1, l_{j_1}^1; k_{i_2}^2, l_{j_2}^2; \dots; k_{i_r}^r, l_{j_r}^r}^{r+1}, \\
&\quad \mu_{k_{i_1}^1, l_{j_1}^1; k_{i_2}^2, l_{j_2}^2; \dots; k_{i_r}^r, l_{j_r}^r}^r, \nu_{k_{i_1}^1, l_{j_1}^1; k_{i_2}^2, l_{j_2}^2; \dots; k_{i_r}^r, l_{j_r}^r}^r \rangle \} \}].
\end{aligned}$$

where

$$\begin{aligned}
\overline{K}_{k_{i_1}^1, k_{i_2}^2, \dots, k_{i_r}^r}^r &= \bigcup_{u=1}^{m_{r+1}} K_{k_{i_1}^1, k_{i_2}^2, \dots, k_{i_r}^r, u}^{r+1}, \\
\overline{L}_{l_{j_1}^1, l_{j_2}^2, \dots, l_{j_r}^r}^r &= \bigcup_{v=1}^{n_{r+1}} L_{l_{j_1}^1, l_{j_2}^2, \dots, l_{j_r}^r, v}^{r+1}
\end{aligned}$$

and for every two $p, q : 1 \leq p < q \leq s$:

$$\begin{aligned}
\overline{K}_{k_{i_1}^1, k_{i_2}^2, \dots, k_{i_p}^r}^p \cap \overline{K}_{l_{j_1}^1, l_{j_2}^2, \dots, l_{j_q}^r}^q &= \emptyset, \\
\overline{L}_{k_{i_1}^1, k_{i_2}^2, \dots, k_{i_p}^r}^p \cap \overline{L}_{l_{j_1}^1, l_{j_2}^2, \dots, l_{j_q}^r}^q &= \emptyset.
\end{aligned}$$

For example, let us have the following two-level HIFIM:

$$\langle A^1, \mu, \nu \rangle = \begin{array}{c|cc} & \delta & \varepsilon \\ \hline \alpha & \langle A_{\alpha, \delta}^2, \mu_{\alpha, \delta}, \nu_{\alpha, \delta} \rangle & \langle A_{\alpha, \varepsilon}^2, \mu_{\alpha, \varepsilon}, \nu_{\alpha, \varepsilon} \rangle \\ \beta & \langle A_{\beta, \delta}^2, \mu_{\beta, \delta}, \nu_{\beta, \delta} \rangle & \langle A_{\beta, \varepsilon}^2, \mu_{\beta, \varepsilon}, \nu_{\beta, \varepsilon} \rangle \\ \gamma & \langle A_{\gamma, \delta}^2, \mu_{\gamma, \delta}, \nu_{\gamma, \delta} \rangle & \langle A_{\gamma, \varepsilon}^2, \mu_{\gamma, \varepsilon}, \nu_{\gamma, \varepsilon} \rangle \end{array},$$

where

$$\langle A_{\alpha, \delta}^2, \mu_{\alpha, \delta}, \nu_{\alpha, \delta} \rangle = \begin{array}{c|cc} & \iota & \kappa \\ \hline \zeta & \langle a, \mu_{\zeta, \iota}, \nu_{\zeta, \iota} \rangle & \langle b, \mu_{\zeta, \kappa}, \nu_{\zeta, \kappa} \rangle \\ \eta & \langle c, \mu_{\eta, \iota}, \nu_{\eta, \iota} \rangle & \langle d, \mu_{\eta, \kappa}, \nu_{\eta, \kappa} \rangle \\ \theta & \langle e, \mu_{\theta, \iota}, \nu_{\theta, \iota} \rangle & \langle f, \mu_{\theta, \kappa}, \nu_{\theta, \kappa} \rangle \end{array},$$

$$\begin{aligned}
\langle A_{\alpha,\varepsilon}^2, \mu_{\alpha,\varepsilon}, \nu_{\alpha,\varepsilon} \rangle &= \frac{\lambda}{\zeta \left| \begin{array}{c} \langle f, \mu_{\zeta,\lambda}, \nu_{\zeta,\lambda} \rangle \end{array} \right|}, \\
\langle A_{\beta,\delta}^2, \mu_{\beta,\delta}, \nu_{\beta,\delta} \rangle &= \frac{\iota}{\xi \left| \begin{array}{c} \langle g, \mu_{\xi,\iota}, \nu_{\xi,\iota} \rangle \langle h, \mu_{\xi,\pi}, \nu_{\xi,\pi} \rangle \end{array} \right|}, \\
\langle A_{\beta,\varepsilon}^2, \mu_{\beta,\varepsilon}, \nu_{\beta,\varepsilon} \rangle &= \frac{\emptyset}{\emptyset \left| \begin{array}{c} \perp \end{array} \right|}, \\
\langle A_{\gamma,\delta}^2, \mu_{\gamma,\delta}, \nu_{\gamma,\delta} \rangle &= \frac{\emptyset}{\emptyset \left| \begin{array}{c} \perp \end{array} \right|}, \\
\langle A_{\gamma,\varepsilon}^2, \mu_{\gamma,\varepsilon}, \nu_{\gamma,\varepsilon} \rangle &= \frac{\lambda}{\theta \left| \begin{array}{c} \langle i, \mu_{\theta,\lambda}, \nu_{\theta,\lambda} \rangle \\ \rho \langle j, \mu_{\rho,\lambda}, \nu_{\rho,\lambda} \rangle \end{array} \right|}.
\end{aligned}$$

Therefore, $\langle A^1, \mu, \nu \rangle$ has the form

$$\langle A^1, \mu, \nu \rangle = \frac{\alpha \left| \begin{array}{c} \delta \quad \varepsilon \\ \langle A_{\alpha,\delta}^2, \mu_{\alpha,\delta}, \nu_{\alpha,\delta} \rangle \quad \langle A_{\alpha,\varepsilon}^2, \mu_{\alpha,\varepsilon}, \nu_{\alpha,\varepsilon} \rangle \\ \beta \langle A_{\beta,\delta}^2, \mu_{\beta,\delta}, \nu_{\beta,\delta} \rangle \quad \perp \\ \gamma \quad \perp \quad \langle A_{\gamma,\varepsilon}^2, \mu_{\gamma,\varepsilon}, \nu_{\gamma,\varepsilon} \rangle \end{array} \right|}{}$$

and (cf. Section 2.6)

$$\begin{aligned}
& (A^1, \mu, \nu)^* \\
= & \begin{array}{c} \zeta \\ \eta \\ \theta \\ \xi \\ \rho \end{array} \left| \begin{array}{cccc} \iota & \kappa & \lambda & \pi \\ \langle a, \mu_{\zeta,\iota}, \nu_{\zeta,\iota} \rangle & \langle b, \mu_{\zeta,\kappa}, \nu_{\zeta,\kappa} \rangle & \langle f, \mu_{\zeta,\lambda}, \nu_{\zeta,\lambda} \rangle & \perp \\ \langle c, \mu_{\eta,\iota}, \nu_{\eta,\iota} \rangle & \langle d, \mu_{\eta,\kappa}, \nu_{\eta,\kappa} \rangle & \perp & \perp \\ \langle e, \mu_{\theta,\iota}, \nu_{\theta,\iota} \rangle & \langle f, \mu_{\theta,\kappa}, \nu_{\theta,\kappa} \rangle & \langle i, \mu_{\theta,\lambda}, \nu_{\theta,\lambda} \rangle & \perp \\ \langle g, \mu_{\xi,\iota}, \nu_{\xi,\iota} \rangle & \perp & \perp & \langle h, \mu_{\xi,\pi}, \nu_{\xi,\pi} \rangle \\ \perp & \perp & \langle j, \mu_{\rho,\lambda}, \nu_{\rho,\lambda} \rangle & \perp \end{array} \right|.
\end{aligned}$$

It is important to mention that some elements of the HIFIM can be variables with their intuitionistic fuzzy evaluations (the degrees of validity and of non-validity). In this case, these elements can be represented as single IMs, as it is the case with $\langle A_{\alpha,\varepsilon}^2, \mu_{\alpha,\varepsilon}, \nu_{\alpha,\varepsilon} \rangle$.

By the moment, we use μ - and ν -functions that determine the degrees of membership and non-membership of the elements of the respective IFIMs. Now, we will describe a method for obtaining of the concrete values of these functions, when we know the concrete values of the functions of the matrices from the lowest level. For this aim, we will use, for example, one of the following formulas, when we have c in number IFPs $\langle a_w, b_w \rangle$, where $1 \leq w \leq c$ and $a_w, b_w, a_w + b_w \in [0, 1]$:

$$\begin{aligned}
\varphi_{\max}(\{\langle a_w, b_w \rangle | 1 \leq w \leq c\}) &= \langle \max_{1 \leq w \leq c} a_w, \min_{1 \leq w \leq c} b_w \rangle, \\
\varphi_{\min}(\{\langle a_w, b_w \rangle | 1 \leq w \leq c\}) &= \langle \min_{1 \leq w \leq c} a_w, \max_{1 \leq w \leq c} b_w \rangle, \\
\varphi_{\text{ave}}(\{\langle a_w, b_w \rangle | 1 \leq w \leq c\}) &= \begin{cases} \langle 0, 0 \rangle, & \text{if } \sum_{1 \leq w \leq c} \text{sg}(a_w + b_w) = 0 \\ \left\langle \frac{\sum_{1 \leq w \leq c} a_w}{\sum_{1 \leq w \leq c} \text{sg}(a_w + b_w)}, \frac{\sum_{1 \leq w \leq c} b_w}{\sum_{1 \leq w \leq c} \text{sg}(a_w + b_w)} \right\rangle, & \text{if } \sum_{1 \leq w \leq c} \text{sg}(a_w + b_w) > 0. \end{cases}
\end{aligned}$$

We start this procedure with the evaluations of the degrees of the lowest level—in our example—the s -th level. There, the IMs will be single ones or IFPs. For example, for the IFIM $\langle A_{\alpha, \varepsilon}^2, \mu_{\alpha, \varepsilon}, \nu_{\alpha, \varepsilon} \rangle$ we will obtain that

$$\langle \mu_{\alpha, \varepsilon}, \nu_{\alpha, \varepsilon} \rangle = \langle \mu_{\zeta, \lambda}, \nu_{\zeta, \lambda} \rangle$$

and we will use element f instead of matrix $A_{\alpha, \varepsilon}^2$.

For matrix $A_{\gamma, \varepsilon}^2$ we can use one of the three above formulas (or another - in the IFS theory there are other similar formulas, too) and will obtain

$$\begin{aligned}
\varphi_{\max}(A_{\gamma, \varepsilon}^2) &= \langle \max(\mu_{\theta, \lambda}, \mu_{\rho, \lambda}), \min(\nu_{\theta, \lambda}, \nu_{\rho, \lambda}) \rangle, \\
\varphi_{\min}(A_{\gamma, \varepsilon}^2) &= \langle \min(\mu_{\theta, \lambda}, \mu_{\rho, \lambda}), \max(\nu_{\theta, \lambda}, \nu_{\rho, \lambda}) \rangle, \\
\varphi_{\text{ave}}(A_{\gamma, \varepsilon}^2) &= \left\langle \frac{\mu_{\theta, \lambda} + \mu_{\rho, \lambda}}{\text{sg}(\mu_{\theta, \lambda} + \nu_{\theta, \lambda}) + \text{sg}(\mu_{\theta, \lambda} + \mu_{\rho, \lambda})}, \frac{\nu_{\theta, \lambda} + \nu_{\rho, \lambda}}{\text{sg}(\mu_{\theta, \lambda} + \nu_{\theta, \lambda}) + \text{sg}(\mu_{\theta, \lambda} + \mu_{\rho, \lambda})} \right\rangle.
\end{aligned}$$

4.4 Extended IFIMs (EIFIMs)

Let I be a fixed set. By EIFIM with index sets K and L ($K, L \subset I$), we denote the object:

$$\begin{aligned}
&[K^*, L^*, \{\langle \mu_{k_i, l_j}, \nu_{k_i, l_j} \rangle\}] \\
&\equiv \begin{array}{c|cccc} & l_1, \langle \alpha_1^L, \beta_1^L \rangle & \dots & l_j, \langle \alpha_j^L, \beta_j^L \rangle & \dots & l_n, \langle \alpha_n^L, \beta_n^L \rangle \\ \hline k_1, \langle \alpha_1^K, \beta_1^K \rangle & \langle \mu_{k_1, l_1}, \nu_{k_1, l_1} \rangle & \dots & \langle \mu_{k_1, l_j}, \nu_{k_1, l_j} \rangle & \dots & \langle \mu_{k_1, l_n}, \nu_{k_1, l_n} \rangle \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_i, \langle \alpha_i^K, \beta_i^K \rangle & \langle \mu_{k_i, l_1}, \nu_{k_i, l_1} \rangle & \dots & \langle \mu_{k_i, l_j}, \nu_{k_i, l_j} \rangle & \dots & \langle \mu_{k_i, l_n}, \nu_{k_i, l_n} \rangle \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_m, \langle \alpha_m^K, \beta_m^K \rangle & \langle \mu_{k_m, l_1}, \nu_{k_m, l_1} \rangle & \dots & \langle \mu_{k_m, l_j}, \nu_{k_m, l_j} \rangle & \dots & \langle \mu_{k_m, l_n}, \nu_{k_m, l_n} \rangle \end{array},
\end{aligned}$$

where $K = \{k_1, k_2, \dots, k_m\}$, $L = \{l_1, l_2, \dots, l_n\}$ and

$$\begin{aligned}
K^* &= \{\langle k_i, \alpha_i^K, \beta_i^K \rangle | k_i \in K\} = \{\langle k_i, \alpha_i^K, \beta_i^K \rangle | 1 \leq i \leq m\}, \\
L^* &= \{\langle l_j, \alpha_j^L, \beta_j^L \rangle | l_j \in L\} = \{\langle l_j, \alpha_j^L, \beta_j^L \rangle | 1 \leq j \leq n\}
\end{aligned}$$

are IFSs and for every $1 \leq i \leq m, 1 \leq j \leq n$:

$$\begin{aligned}\mu_{k_i, l_j}, \nu_{k_i, l_j}, \mu_{k_i, l_j} + \nu_{k_i, l_j} &\in [0, 1], \\ \alpha_i^K, \beta_i^K, \alpha_i^K + \beta_i^K &\in [0, 1], \\ \alpha_j^L, \beta_j^L, \alpha_j^L + \beta_j^L &\in [0, 1].\end{aligned}$$

First, we will describe the standard forms of the operations over EIFIMs that are analogous of these from Chapter 1 and Section 4.2.

For the EIFIMs

$$\begin{aligned}A &= [K^*, L^*, \{\langle \mu_{k_i, l_j}, \nu_{k_i, l_j} \rangle\}], \\ B &= [P^*, Q^*, \{\langle \rho_{p_r, q_s}, \sigma_{p_r, q_s} \rangle\}],\end{aligned}$$

we will introduce only these operations, relations and operators that are necessary for the current research.

Addition-(max,min) or Addition-($\vee$)

$$A \oplus_{(\max, \min)} B = [T^*, V^*, \{\langle \varphi_{t_u, v_w}, \psi_{t_u, v_w} \rangle\}],$$

where

$$\begin{aligned}T^* &= K^* \cup P^* = \{\langle t_u, \langle \alpha_u^K, \beta_u^K \rangle \vee \langle \alpha_u^P, \beta_u^P \rangle \rangle | t_u \in K \cup P\}, \\ &= \{\langle t_u, \langle \alpha_u^T, \beta_u^T \rangle \rangle | t_u \in K \cup P\} \\ V^* &= L^* \cup Q^* = \{\langle v_w, \langle \alpha_w^L, \beta_w^L \rangle \vee \langle \alpha_w^Q, \beta_w^Q \rangle \rangle | v_w \in L \cup Q\}, \\ &= \{\langle v_w, \langle \alpha_w^V, \beta_w^V \rangle \rangle | v_w \in L \cup Q\},\end{aligned}$$

where

$$\alpha_u^T = \begin{cases} \alpha_i^K, & \text{if } t_u \in K - P \\ \alpha_r^P, & \text{if } t_u \in P - K, \\ \max(\alpha_i^K, \alpha_r^P), & \text{if } t_u \in K \cap P \end{cases}, \quad \beta_w^V = \begin{cases} \beta_j^L, & \text{if } v_w \in L - Q \\ \beta_s^Q, & \text{if } v_w \in Q - L, \\ \min(\beta_j^L, \beta_s^Q), & \text{if } v_w \in L \cap Q \end{cases}$$

$$\begin{aligned} &\langle \varphi_{t_u, v_w}, \psi_{t_u, v_w} \rangle \\ = &\begin{cases} \langle \mu_{k_i, l_j}, \nu_{k_i, l_j} \rangle, & \text{if } t_u = k_i \in K \text{ and } v_w = l_j \in L - Q \\ & \text{or } t_u = k_i \in K - P \text{ and } v_w = l_j \in L; \\ \langle \rho_{p_r, q_s}, \sigma_{p_r, q_s} \rangle, & \text{if } t_u = p_r \in P \text{ and } v_w = q_s \in Q - L \\ & \text{or } t_u = p_r \in P - K \text{ and } v_w = q_s \in Q; \\ \langle \max(\mu_{k_i, l_j}, \rho_{p_r, q_s}), \min(\nu_{k_i, l_j}, \sigma_{p_r, q_s}) \rangle, & \text{if } t_u = k_i = p_r \in K \cap P \\ & \text{and } v_w = l_j = q_s \in L \cap Q; \\ \langle 0, 1 \rangle, & \text{otherwise.} \end{cases}\end{aligned}$$

Addition-(min,max) or Addition-($\wedge$)

$$A \oplus_{(\min, \max)} B = [T^*, V^*, \{\langle \varphi_{t_u, v_w}, \psi_{t_u, v_w} \rangle\}],$$

where $T^*, V^*, \alpha_u^T, \beta_w^V$, have the above forms and

$$\langle \varphi_{t_u, v_w}, \psi_{t_u, v_w} \rangle = \begin{cases} \langle \mu_{k_i, l_j}, \nu_{k_i, l_j} \rangle, & \text{if } t_u = k_i \in K \text{ and } v_w = l_j \in L - Q \\ & \text{or } t_u = k_i \in K - P \text{ and } v_w = l_j \in L; \\ \langle \rho_{p_r, q_s}, \sigma_{p_r, q_s} \rangle, & \text{if } t_u = p_r \in P \text{ and } v_w = q_s \in Q - L \\ & \text{or } t_u = p_r \in P - K \text{ and } v_w = q_s \in Q; \\ \langle \min(\mu_{k_i, l_j}, \rho_{p_r, q_s}), \max(\nu_{k_i, l_j}, \sigma_{p_r, q_s}) \rangle, & \text{if } t_u = k_i = p_r \in K \cap P \\ & \text{and } v_w = l_j = q_s \in L \cap Q; \\ \langle 0, 1 \rangle, & \text{otherwise.} \end{cases}$$

Termwise multiplication-(max,min) or Termwise multiplication-($\vee$)

$$A \otimes_{(\max, \min)} B = [T^*, V^*, \{\langle \varphi_{t_u, v_w}, \psi_{t_u, v_w} \rangle\}],$$

where

$$T^* = K^* \cap P^* = \{ \langle t_u, \langle \alpha_u^K, \beta_u^K \rangle \& \langle \alpha_u^P, \beta_u^P \rangle \rangle | t_u \in K \cup P \},$$

$$= \{ \langle t_u, \langle \alpha_u^T, \beta_u^T \rangle \rangle | t_u \in K \cup P \}$$

$$V^* = L^* \cap Q^* = \{ \langle v_w, \langle \alpha_w^L, \beta_w^L \rangle \& \langle \alpha_w^Q, \beta_w^Q \rangle \rangle | v_w \in L \cup Q \},$$

$$= \{ \langle v_w, \langle \alpha_w^V, \beta_w^V \rangle \rangle | v_w \in L \cup Q \},$$

$$\alpha_u^T = \min(\alpha_i^K, \alpha_r^P), \text{ for } t_u = k_i = p_r \in K \cap P,$$

$$\beta_w^V = \min(\beta_j^L, \beta_s^Q), \text{ for } v_w = l_j = q_s \in L \cap Q$$

and

$$\langle \varphi_{t_u, v_w}, \psi_{t_u, v_w} \rangle = \langle \max(\mu_{k_i, l_j}, \rho_{p_r, q_s}), \min(\nu_{k_i, l_j}, \sigma_{p_r, q_s}) \rangle.$$

Termwise multiplication-(min,max) or Termwise multiplication-($\wedge$)

$$A \otimes_{(\min, \max)} B = [T^*, V^*, \{\langle \varphi_{t_u, v_w}, \psi_{t_u, v_w} \rangle\}],$$

where $T^*, V^*, \alpha_u^T, \beta_w^V$, have the above forms and

$$\langle \varphi_{t_u, v_w}, \psi_{t_u, v_w} \rangle = \langle \min(\mu_{k_i, l_j}, \rho_{p_r, q_s}), \max(\nu_{k_i, l_j}, \sigma_{p_r, q_s}) \rangle.$$

Second, following [38], we will describe extensions of the above operations, that are specific for EIFIMs.

Let everywhere below $\circ, * \in \{\vee, \wedge\}$ and let when $\circ$ or $*$ be $\vee$, then $\circ_1$ or $*_1$ be “max”, and $\circ_2$ or $*_2$ be “min”; and when $\circ$ or $*$ be $\wedge$, then $\circ_1$ or $*_1$ be “min”, and $\circ_2$ or $*_2$ be “max”. Let $e_\vee = \langle 0, 1 \rangle$ and $e_\wedge = \langle 1, 0 \rangle$.

For the EIFIMs A and B operations that are analogous to the standard matrix operations of addition and multiplication are defined, as well as other specific ones.

Addition $\oplus^*$

$$A \oplus_\circ^* B = [T^*, V^*, \{\langle \varphi_{t_u, v_w}, \psi_{t_u, v_w} \rangle\}],$$

where

$$T^* = K^* \cup P^* = \{\langle t_u, \alpha_u^t, \beta_u^t \rangle | t_u \in K \cup P\},$$

$$V^* = L^* \cup Q^* = \{\langle v_w, \alpha_w^v, \beta_w^v \rangle | v_w \in L \cup Q\},$$

$$\alpha_u^t = \begin{cases} \alpha_i^k, & \text{if } t_u \in K - P \\ \alpha_r^p, & \text{if } t_u \in P - K, \\ *_1(\alpha_i^k, \alpha_r^p), & \text{if } t_u \in K \cap P \end{cases} \quad \beta_w^v = \begin{cases} \beta_j^l, & \text{if } v_w \in L - Q \\ \beta_s^q, & \text{if } v_w \in Q - L, \\ *_2(\beta_j^l, \beta_s^q), & \text{if } v_w \in L \cap Q \end{cases}$$

and

$$\langle \varphi_{t_u, v_w}, \psi_{t_u, v_w} \rangle = \begin{cases} \langle \mu_{k_i, l_j}, \nu_{k_i, l_j} \rangle, & \text{if } t_u = k_i \in K \text{ and } v_w = l_j \in L - Q \\ & \text{or } t_u = k_i \in K - P \text{ and } v_w = l_j \in L; \\ \langle \sigma_{p_r, q_s}, \tau_{p_r, q_s} \rangle, & \text{if } t_u = p_r \in P \text{ and } v_w = q_s \in Q - L \\ & \text{or } t_u = p_r \in P - K \text{ and } v_w = q_s \in Q; \\ \langle \circ_1(\mu_{k_i, l_j}, \sigma_{p_r, q_s}), \circ_2(\nu_{k_i, l_j}, \tau_{p_r, q_s}) \rangle, & \text{if } t_u = k_i = p_r \in K \cap P \\ & \text{and } v_w = l_j = q_s \in L \cap Q; \\ e_\circ, & \text{otherwise.} \end{cases}$$

Termwise multiplication $\otimes^*$

$$A \otimes_\circ^* B = [T^*, V^*, \{\langle \varphi_{t_u, v_w}, \psi_{t_u, v_w} \rangle\}],$$

where

$$T^* = K^* \cap P^* = \{\langle t_u, \alpha_u^t, \beta_u^t \rangle | t_u \in K \cap P\},$$

$$V^* = L^* \cap Q^* = \{\langle v_w, \alpha_w^v, \beta_w^v \rangle | v_w \in L \cap Q\},$$

$$\alpha_u^t = *_1(\alpha_i^k, \alpha_r^p), \text{ for } t_u = k_i = p_r \in K \cap P,$$

$$\beta_w^v = *_2(\beta_j^l, \beta_s^q), \text{ for } v_w = l_j = q_s \in L \cap Q$$

and

$$\langle \varphi_{t_u, v_w}, \psi_{t_u, v_w} \rangle = \langle \circ_1(\mu_{k_i, l_j}, \sigma_{p_r, q_s}), \circ_2(\nu_{k_i, l_j}, \tau_{p_r, q_s}) \rangle.$$

Multiplication

$$A \odot_{(\circ,*)} B = [T^*, V^*, \langle \varphi_{t_u, v_w}, \psi_{t_u, v_w} \rangle],$$

where

$$\begin{aligned} T^* &= (K \cup (P - L))^* = \{\langle t_u, \alpha_u^t, \beta_u^t \rangle | t_u \in K \cup (P - L)\}, \\ V^* &= (Q \cup (L - P))^* = \{\langle v_w, \alpha_w^v, \beta_w^v \rangle | v_w \in Q \cup (L - P)\}, \end{aligned}$$

$$\alpha_u^t = \begin{cases} \alpha_i^k, & \text{if } t_u = k_i \in K; \\ \alpha_r^p, & \text{if } t_u = p_r \in P - L, \end{cases}, \quad \beta_w^v = \begin{cases} \beta_j^l, & \text{if } v_w = l_j \in L - P; \\ \beta_s^q, & \text{if } v_w = q_s \in Q \end{cases},$$

and

$$\begin{aligned} & \langle \varphi_{t_u, v_w}, \psi_{t_u, v_w} \rangle \\ = & \begin{cases} \langle \mu_{k_i, l_j}, \nu_{k_i, l_j} \rangle, & \text{if } t_u = k_i \in K \\ & \text{and } v_w = l_j \in L - P - Q; \\ \langle \sigma_{p_r, q_s}, \tau_{p_r, q_s} \rangle, & \text{if } t_u = p_r \in P - L - K \\ & \text{and } v_w = q_s \in Q; \\ \left\langle \begin{aligned} & \underset{l_j = p_r \in L \cap P}{\circ_1} (*_1(\mu_{k_i, l_j}, \sigma_{p_r, q_s})) , & \text{if } t_u = k_i \in K \text{ and } v_w = q_s \in Q; \\ & \underset{l_j = p_r \in L \cap P}{\circ_2} (*_2(\nu_{k_i, l_j}, \tau_{p_r, q_s})) \end{aligned} \right\rangle, \\ e_\circ, & \text{otherwise.} \end{cases} \end{aligned}$$

Third, by analogy with the operation “substitution” described in Section 2.3, we introduce the same operation over EIFIMs (cf. [38, 39]). The new operation is specific only for this type of IMs.

Let the EIFIM A be given and let below $\perp$ denote the lack of components.

A local substitution of an index degree of the EIFIM is defined for the index $k \in K$ or $l \in L$, respectively, by

$$\begin{aligned} & \left[\frac{\langle k, \gamma^k, \delta^k \rangle}{\langle k, \alpha^k, \beta^k \rangle}; \perp \right] A \\ &= [(K^* - \{\langle k, \alpha^k, \beta^k \rangle\}) \cup \{\langle k, \gamma^k, \delta^k \rangle\}, L, \{\langle a_{k, l}, \mu_{k_i, l_j}, \nu_{k_i, l_j} \rangle\}], \\ & \left[\perp; \frac{\langle l, \gamma^l, \delta^l \rangle}{\langle l, \alpha^l, \beta^l \rangle} \right] A \\ &= [K^*, (L^* - \{\langle l, \alpha^l, \beta^l \rangle\}) \cup \{\langle l, \gamma^l, \delta^l \rangle\}, L, \{\langle a_{l, l}, \mu_{l_i, l_j}, \nu_{l_i, l_j} \rangle\}], \\ & \left[\frac{\langle k, \gamma^k, \delta^k \rangle}{\langle k, \alpha^k, \beta^k \rangle}; \frac{\langle l, \gamma^l, \delta^l \rangle}{\langle l, \alpha^l, \beta^l \rangle} \right] A = \left[\frac{\langle k, \gamma^k, \delta^k \rangle}{\langle k, \alpha^k, \beta^k \rangle}; \perp \right] \left(\left[\perp; \frac{\langle l, \gamma^l, \delta^l \rangle}{\langle l, \alpha^l, \beta^l \rangle} \right] A \right) \\ &= \left[\perp; \frac{\langle l, \gamma^l, \delta^l \rangle}{\langle l, \alpha^l, \beta^l \rangle} \right] \left(\left[\frac{\langle k, \gamma^k, \delta^k \rangle}{\langle k, \alpha^k, \beta^k \rangle}; \perp \right] A \right). \end{aligned}$$

A local substitution of an index together with its degrees of the EIFIM is defined for the pairs of indices (p, k) and/or (q, l) , respectively, by

$$\begin{aligned}
& \left[\frac{\langle p, \alpha^p, \beta^p \rangle}{\langle k, \alpha^k, \beta^k \rangle}; \perp \right] A \\
&= \left[(K^* - \{\langle k, \alpha^k, \beta^k \rangle\}) \cup \{\langle p, \alpha^p, \beta^p \rangle\}, L, \{\langle a_{k,l}, \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle\} \right], \\
& \left[\perp; \frac{\langle q, \alpha^q, \beta^q \rangle}{\langle l, \alpha^l, \beta^l \rangle} \right] A \\
&= \left[K^*, (L^* - \{\langle l, \alpha^l, \beta^l \rangle\}) \cup \{\langle q, \alpha^q, \beta^q \rangle\}, L, \{\langle a_{l,l}, \mu_{l_i,l_j}, \nu_{l_i,l_j} \rangle\} \right], \\
& \left[\frac{\langle p, \alpha^p, \beta^p \rangle}{\langle k, \alpha^k, \beta^k \rangle}; \frac{\langle q, \alpha^q, \beta^q \rangle}{\langle l, \alpha^l, \beta^l \rangle} \right] A = \left[\frac{\langle p, \alpha^p, \beta^p \rangle}{\langle k, \alpha^k, \beta^k \rangle}; \perp \right] \left(\left[\perp; \frac{\langle q, \alpha^q, \beta^q \rangle}{\langle l, \alpha^l, \beta^l \rangle} \right] A \right) \\
&= \left[\perp; \frac{\langle q, \alpha^q, \beta^q \rangle}{\langle l, \alpha^l, \beta^l \rangle} \right] \left(\left[\frac{\langle p, \alpha^p, \beta^p \rangle}{\langle k, \alpha^k, \beta^k \rangle}; \perp \right] A \right).
\end{aligned}$$

Obviously, for the above indices k, l, p, q :¹

$$\begin{aligned}
& \left[\frac{\langle k, \alpha^k, \beta^k \rangle}{\langle p, \alpha^p, \beta^p \rangle}; \perp \right] \left(\left[\frac{\langle p, \alpha^p, \beta^p \rangle}{\langle k, \alpha^k, \beta^k \rangle}; \perp \right] A \right) = \left[\perp; \frac{\langle l, \alpha^l, \beta^l \rangle}{\langle q, \alpha^q, \beta^q \rangle} \right] \left(\left[\perp; \frac{\langle q, \alpha^q, \beta^q \rangle}{\langle l, \alpha^l, \beta^l \rangle} \right] A \right), \\
& \left[\frac{\langle k, \alpha^k, \beta^k \rangle}{\langle p, \alpha^p, \beta^p \rangle}; \frac{\langle l, \alpha^l, \beta^l \rangle}{\langle q, \alpha^q, \beta^q \rangle} \right] \left(\left[\frac{\langle p, \alpha^p, \beta^p \rangle}{\langle k, \alpha^k, \beta^k \rangle}; \frac{\langle q, \alpha^q, \beta^q \rangle}{\langle l, \alpha^l, \beta^l \rangle} \right] A \right) = A.
\end{aligned}$$

Let the sets of indices $P = \{p_1, p_2, \dots, p_u\}$ and $Q = \{q_1, q_2, \dots, q_v\}$ be given. For them we define sequentially:

$$\begin{aligned}
& \left[\frac{P^*}{K^*}; \perp \right] A = \left[\frac{\langle p_1, \alpha^{p_1}, \beta^{p_1} \rangle}{\langle k_1, \alpha^{k_1}, \beta^{k_1} \rangle} \dots \frac{\langle p_u, \alpha^{p_u}, \beta^{p_u} \rangle}{\langle k_u, \alpha^{k_u}, \beta^{k_u} \rangle}; \perp \right] A, \\
& \left[\perp; \frac{Q^*}{L^*} \right] A = \left[\perp; \frac{\langle q_1, \alpha^{q_1}, \beta^{q_1} \rangle}{\langle l_1, \alpha^{l_1}, \beta^{l_1} \rangle} \dots \frac{\langle q_v, \alpha^{q_v}, \beta^{q_v} \rangle}{\langle l_v, \alpha^{l_v}, \beta^{l_v} \rangle} \right] A, \\
& \left[\frac{P^*}{K^*}; \frac{Q^*}{L^*} \right] A = \left[\frac{P^*}{K^*}; \perp \right] \left(\left[\perp; \frac{Q^*}{L^*} \right] A \right).
\end{aligned}$$

Obviously, for the sets K, L, P, Q :

$$\begin{aligned}
& \left[\frac{K^*}{P^*}; \perp \right] \left(\left[\frac{P^*}{K^*}; \perp \right] A \right) = \left[\perp; \frac{L^*}{Q^*} \right] \left(\left[\perp; \frac{Q^*}{L^*} \right] A \right) \\
&= \left[\frac{K^*}{P^*}; \frac{L^*}{Q^*} \right] \left(\left[\frac{P^*}{K^*}; \frac{Q^*}{L^*} \right] A \right) = A.
\end{aligned}$$

¹ In the analogical formulas in [20] there are some misprints which are here corrected.

From the above definitions, in [38] it is proved that for every four sets of indices $P_1^*, P_2^*, Q_1^*, Q_2^*$

$$\left[\frac{P_2^*}{P_1^*}, \frac{Q_2^*}{Q_1^*} \right] \left[\frac{P_1^*}{K^*}, \frac{Q_1^*}{L^*} \right] A = \left[\frac{P_2^*}{K^*}, \frac{Q_2^*}{L^*} \right] A.$$

Finally, for $k \in K$ and $l \in L$ and for an element $\langle \mu_{k,l}, \nu_{k,l} \rangle$ of the EIFIM A , we define the substitution

$$\left\{ \frac{\langle k, l \rangle}{\langle \zeta, \eta \rangle} \right\} A$$

and as a result we obtain the EIFIM that has the element $\langle \zeta, \eta \rangle$ in its k -th row and l -th column instead of the element $\langle \mu_{k,l}, \nu_{k,l} \rangle$.

Obviously,

$$\left\{ \frac{\langle k, l \rangle}{\langle \mu_{k,l}, \nu_{k,l} \rangle} \right\} \left(\left\{ \frac{\langle k, l \rangle}{\langle \zeta, \eta \rangle} \right\} A \right) = A.$$

The operations “projection” and “reduction” are the same as in Chapter 1.

Fourth, let $(\circ, *) \in \{(\max, \min), (\min, \max), +, \cdot, \dots\}$. In [68], over the EIFIMs A and B the operation $\bigotimes$ is defined in the form $\bigotimes_{(\circ_1, *_1); (\circ_2, *_2)}$, by:

$$A \bigotimes_{(\circ, *)} B = [T^*, V^*, \{\langle c_{t_u, v_w}, d_{t_u, v_w} \rangle\}],$$

where

$$\begin{aligned} T^* &= K^* \cup P^* = \{\langle t_u, \alpha_u^t, \beta_u^t \rangle \mid t_u \in K \cup P\}, \\ V^* &= L^* \cup Q^* = \{\langle v_w, \alpha_w^v, \beta_w^v \rangle \mid v_w \in L \cup Q\}, \end{aligned}$$

and

$$\begin{aligned} \langle \alpha_u^t, \beta_u^t \rangle &= \begin{cases} \langle \alpha_i^k, \beta_i^k \rangle, & \text{if } t_u = k_i \in K - P; \\ \langle \alpha_r^p, \beta_r^p \rangle, & \text{if } t_u = p_r \in P - K; \\ \langle \circ(\alpha_i^k, \alpha_r^p), *(\beta_i^k, \beta_r^p) \rangle & \text{if } t_u = k_i = p_r \in K \cap P, \end{cases} \\ \langle \alpha_w^v, \beta_w^v \rangle &= \begin{cases} \langle \alpha_j^l, \beta_j^l \rangle, & \text{if } v_w = l_j \in L - Q; \\ \langle \alpha_s^q, \beta_s^q \rangle, & \text{if } v_w = q_s \in Q - L; \\ \langle \circ(\alpha_j^l, \alpha_s^q), *(\beta_j^l, \beta_s^q) \rangle; & \text{if } v_w = l_j = q_s \in L \cap Q, \end{cases} \end{aligned}$$

and

$$= \begin{cases} \langle c_{t_u, v_w}, d_{t_u, v_w} \rangle & \\ \langle \langle \mu_{k_i, l_j}, \nu_{k_i, l_j} \rangle, \perp \rangle, & \text{if } t_u = k_i \in K \text{ and } v_w = l_j \in L - Q \\ & \text{or } t_u = k_i \in K - P \text{ and } v_w = l_j \in L; \\ \langle \perp, \langle \mu_{p_r, q_s}, \nu_{p_r, q_s} \rangle \rangle, & \text{if } t_u = p_r \in P \text{ and } v_w = q_s \in Q - L \\ & \text{or } t_u = p_r \in P - K \text{ and } v_w = q_s \in Q; \\ \langle \langle \mu_{k_i, l_j}, \mu_{p_r, q_s} \rangle, \langle \nu_{k_i, l_j}, \nu_{p_r, q_s} \rangle \rangle, & \text{if } t_u = k_i = p_r \in K \cap P \\ & \text{and } v_w = l_j = q_s \in L \cap Q; \\ \langle \perp, \perp \rangle. & \text{otherwise} \end{cases}$$

In the partial case, when A and B are IFIMs,

$$A \otimes_{(\perp, \perp)} B = [T, V, \{\langle c_{t_u, v_w}, d_{t_u, v_w} \rangle\}],$$

where $T = K \cup P$, $V = L \cup Q$, and

$$= \begin{cases} \langle c_{t_u, v_w}, d_{t_u, v_w} \rangle & \\ \langle \langle \mu_{k_i, l_j}, \nu_{k_i, l_j} \rangle, \perp \rangle, & \text{if } t_u = k_i \in K \text{ and } v_w = l_j \in L - Q \\ & \text{or } t_u = k_i \in K - P \text{ and } v_w = l_j \in L; \\ \langle \perp, \langle \mu_{p_r, q_s}, \nu_{p_r, q_s} \rangle \rangle, & \text{if } t_u = p_r \in P \text{ and } v_w = q_s \in Q - L \\ & \text{or } t_u = p_r \in P - K \text{ and } v_w = q_s \in Q; \\ \langle \langle \mu_{k_i, l_j}, \mu_{p_r, q_s} \rangle, \langle \nu_{k_i, l_j}, \nu_{p_r, q_s} \rangle \rangle, & \text{if } t_u = k_i = p_r \in K \cap P \\ & \text{and } v_w = l_j = q_s \in L \cap Q; \\ \langle \perp, \perp \rangle, & \text{otherwise.} \end{cases}$$

For brevity, we can write $A \otimes B = A \otimes_{(\perp, \perp)} B$.

Now, we can check that operation $\otimes$ over EIFIMs is not idempotent and commutative, while it is associative.

Let $\mathcal{M}$ be the set of all EIFIMs. Let

$$I_\emptyset = [\emptyset, \emptyset, \perp],$$

where symbol “ $\perp$ ” here denotes the lack of IM-elements.

Let for every natural number $n \geq 2$ and for every n EIFIMs $A_1, \dots, A_n$ such that for s ($1 \leq s \leq n$):

$$A_s = \begin{array}{c|ccc} & l_{s,1}, \langle \alpha_{s,1}^l, \beta_{s,1}^l \rangle & \dots & l_{s,n_s}, \langle \alpha_{s,n_s}^l, \beta_{s,n_s}^l \rangle \\ \hline k_{s,1}, \langle \alpha_{s,1}^k, \beta_{s,1}^k \rangle & \langle \mu_{k_{s,1}, l_{s,1}}, \nu_{k_{s,1}, l_{s,1}} \rangle & \dots & \langle \mu_{k_{s,1}, l_{s,n_s}}, \nu_{k_{s,1}, l_{s,n_s}} \rangle \\ \vdots & \vdots & \ddots & \vdots \\ k_{s,i}, \langle \alpha_{s,i}^k, \beta_{s,i}^k \rangle & \langle \mu_{k_{s,i}, l_{s,1}}, \nu_{k_{s,i}, l_{s,1}} \rangle & \dots & \langle \mu_{k_{s,i}, l_{s,n_s}}, \nu_{k_{s,i}, l_{s,n_s}} \rangle \\ \vdots & \vdots & \ddots & \vdots \\ k_{s,m_s}, \langle \alpha_{s,m_s}^k, \beta_{s,m_s}^k \rangle & \langle \mu_{k_{s,m_s}, l_{s,1}}, \nu_{k_{s,m_s}, l_{s,1}} \rangle & \dots & \langle \mu_{k_{s,m_s}, l_{s,n_s}}, \nu_{k_{s,m_s}, l_{s,n_s}} \rangle \end{array},$$

it holds true that

$$\bigotimes_{i=1}^2 A_i = A_1 \otimes A_2,$$

$$\bigotimes_{i=1}^n A_i = \left(\bigotimes_{i=1}^{n-1} A_i \right) \otimes A_n.$$

We can mention that in [68], it is proved that $\langle \mathcal{M}, \otimes, I_\emptyset \rangle$ is a monoid.

Fifth, let

$$K^* \subset P^* \text{ iff } (K \subset P) \ \& \ (\forall k_i = p_i \in K)((\alpha_i^K < \alpha_i^P) \ \& \ (\beta_i^K > \beta_i^P)),$$

$$K^* \subseteq P^* \text{ iff } (K \subseteq P) \ \& \ (\forall k_i = p_i \in K)((\alpha_i^K \leq \alpha_i^P) \ \& \ (\beta_i^K \geq \beta_i^P)).$$

Let the two EIFIMs A and B be given. We shall introduce the following definitions for relations over EIFIMs, where $\subset$ and $\subseteq$ denote the relations “*strong inclusion*” and “*weak inclusion*”, respectively. We must mention that in the present form, they are new and extend the existing definitions given in [20].

The strict relation “inclusion about dimension” is

$$A \subset_d B \text{ iff } (((K^* \subset P^*) \ \& \ (L^* \subset Q^*)) \vee ((K^* \subseteq P^*) \ \& \ (L^* \subset Q^*)))$$

$$\vee ((K^* \subset P^*) \ \& \ (L^* \subseteq Q^*))$$

$$\ \& \ (\forall k \in K)(\forall l \in L)((\langle \alpha_k^K, \beta_k^K \rangle = \langle \alpha_k^P, \beta_k^P \rangle)$$

$$\ \& \ (\langle \alpha_l^L, \beta_l^L \rangle = \langle \alpha_l^Q, \beta_l^Q \rangle) \ \& \ (\langle \mu_{k,l}, \nu_{k,l} \rangle = \langle \rho_{k,l}, \sigma_{k,l} \rangle)).$$

The non-strict relation “inclusion about dimension” is

$$A \subseteq_d B \text{ iff } (K^* \subseteq P^*) \ \& \ (L^* \subseteq Q^*) \ \& \ (\forall k \in K)(\forall l \in L)$$

$$((\langle \alpha_k^K, \beta_k^K \rangle = \langle \alpha_k^P, \beta_k^P \rangle) \ \& \ (\langle \alpha_l^L, \beta_l^L \rangle = \langle \alpha_l^Q, \beta_l^Q \rangle))$$

$$\ \& \ (\langle \mu_{k,l}, \nu_{k,l} \rangle = \langle \rho_{k,l}, \sigma_{k,l} \rangle)).$$

The strict relation “inclusion about element values” is

$$A \subset_v B \text{ iff } (K^* = P^*) \ \& \ (L^* = Q^*) \ \& \ (\forall k \in K)(\forall l \in L)$$

$$((\langle \alpha_k^K, \beta_k^K \rangle = \langle \alpha_k^P, \beta_k^P \rangle) \ \& \ (\langle \alpha_l^L, \beta_l^L \rangle = \langle \alpha_l^Q, \beta_l^Q \rangle))$$

$$\ \& \ (\langle \mu_{k,l}, \nu_{k,l} \rangle < \langle \rho_{k,l}, \sigma_{k,l} \rangle)).$$

The non-strict relation “inclusion about element values” is

$$A \subseteq_v B \text{ iff } (K^* = P^*) \ \& \ (L^* = Q^*) \ \& \ (\forall k \in K)(\forall l \in L)$$

$$((\langle \alpha_k^K, \beta_k^K \rangle = \langle \alpha_k^P, \beta_k^P \rangle) \ \& \ (\langle \mu_{k,l}, \nu_{k,l} \rangle \leq \langle \rho_{k,l}, \sigma_{k,l} \rangle)).$$

The strict relation “inclusion about indices” is relation !strict inclusion about indices

$$\begin{aligned} A \subset_i B \text{ iff } & (K^* = P^*) \ \& \ (L^* = Q^*) \ \& \ (\forall k \in K)(\forall l \in L) \\ & ((\langle \alpha_k^K, \beta_k^K \rangle < \langle \alpha_k^P, \beta_k^P \rangle) \ \& \ (\langle \alpha_l^L, \beta_l^L \rangle < \langle \alpha_l^Q, \beta_l^Q \rangle)) \\ & \ \& \ (\langle \mu_{k,l}, \nu_{k,l} \rangle = \langle \rho_{k,l}, \sigma_{k,l} \rangle). \end{aligned}$$

The non-strict relation “inclusion about indices” is

$$\begin{aligned} A \subseteq_i B \text{ iff } & (K^* = P^*) \ \& \ (L^* = Q^*) \ \& \ (\forall k \in K)(\forall l \in L) \\ & ((\langle \alpha_k^K, \beta_k^K \rangle \leq \langle \alpha_k^P, \beta_k^P \rangle) \ \& \ (\langle \alpha_l^L, \beta_l^L \rangle \leq \langle \alpha_l^Q, \beta_l^Q \rangle)) \\ & \ \& \ (\langle \mu_{k,l}, \nu_{k,l} \rangle = \langle \rho_{k,l}, \sigma_{k,l} \rangle). \end{aligned}$$

We can also introduce combinations of these relations in the following forms:

$$\begin{aligned} A \subset_{d,v} B \text{ iff } & (A \subset_d B) \ \& \ (A \subset_v B), \\ A \subseteq_{d,v} B \text{ iff } & (A \subseteq_d B) \ \& \ (A \subseteq_v B), \\ A \subset_{d,i} B \text{ iff } & (A \subset_d B) \ \& \ (A \subset_i B), \\ A \subseteq_{d,i} B \text{ iff } & (A \subseteq_d B) \ \& \ (A \subseteq_i B), \\ A \subset_{v,i} B \text{ iff } & (A \subset_v B) \ \& \ (A \subseteq_i B), \\ A \subseteq_{v,i} B \text{ iff } & (A \subseteq_v B) \ \& \ (A \subseteq_i B), \\ A \subset B \text{ iff } & (A \subset_d B) \ \& \ (A \subset_v B) \ \& \ (A \subset_i B), \\ A \subseteq B \text{ iff } & (A \subseteq_d B) \ \& \ (A \subseteq_v B) \ \& \ (A \subseteq_i B). \end{aligned}$$

The strict relation “inclusion about first index values” is

$$A \subset_{i,f} B \text{ iff } (K^* = P^*) \ \& \ (\forall k_i = p_q \in K)((\alpha_i^k < \alpha_q^p) \ \& \ (\beta_i^k > \beta_q^p)).$$

The non-strict relation “inclusion about first index values” is

$$A \subseteq_{i,f} B \text{ iff } (K^* = P^*) \ \& \ (\forall k_i = p_q \in K)((\alpha_i^k \leq \alpha_q^p) \ \& \ (\beta_i^k \geq \beta_q^p)).$$

The strict relation “inclusion about second index values” is

$$A \subset_{i,s} B \text{ iff } (L^* = Q^*) \ \& \ (\forall l_j = r_s \in L)((\alpha_j^l < \alpha_s^r) \ \& \ (\beta_j^l > \beta_s^r)).$$

The non-strict relation “inclusion about first index values” is

$$A \subseteq_{i,s} B \text{ iff } (L^* = Q^*) \ \& \ (\forall l_j = r_s \in L)((\alpha_j^l \leq \alpha_s^r) \ \& \ (\beta_j^l \geq \beta_s^r)).$$

4.5 Operators over IFIMs and EIFIMs

Let $A = [K^*, L^*, \{\langle \mu_{k_i, l_j}, \nu_{k_i, l_j} \rangle\}]$ be an EIFIM, where the sets K^* and L^* are defined above. Let $\mathcal{O}_{\varphi_1, \psi_1}^1$, $\mathcal{O}_{\varphi_2, \psi_2}^2$ and $\mathcal{O}_{\varphi_3, \psi_3}^3$ be three of the operators defined in Section 4.1 ($\square, \diamond, D_\alpha, \dots, Q_{\alpha, \beta}$). Then

$$\begin{aligned}
 & (\mathcal{O}_{\varphi_1, \psi_1}^1, \perp, \perp)A \\
 &= [\mathcal{O}_{\varphi_1, \psi_1}^1(K^*), L^*, \{\langle \mu_{k_i, l_j}, \nu_{k_i, l_j} \rangle\}] \\
 &= \begin{array}{c|ccc} & l_1, \langle \alpha_1^l, \beta_1^l \rangle & \dots & l_n, \langle \alpha_n^l, \beta_n^l \rangle \\ \hline k_1, \mathcal{O}_{\varphi_1, \psi_1}^1(\langle \alpha_1^k, \beta_1^k \rangle) & \langle \mu_{k_1, l_1}, \nu_{k_1, l_1} \rangle & \dots & \langle \mu_{k_1, l_n}, \nu_{k_1, l_n} \rangle \\ \vdots & \vdots & \ddots & \vdots \\ k_i, \mathcal{O}_{\varphi_1, \psi_1}^1(\langle \alpha_i^k, \beta_i^k \rangle) & \langle \mu_{k_i, l_1}, \nu_{k_i, l_1} \rangle & \dots & \langle \mu_{k_i, l_n}, \nu_{k_i, l_n} \rangle \\ \vdots & \vdots & \ddots & \vdots \\ k_m, \mathcal{O}_{\varphi_1, \psi_1}^1(\langle \alpha_m^k, \beta_m^k \rangle) & \langle \mu_{k_m, l_1}, \nu_{k_m, l_1} \rangle & \dots & \langle \mu_{k_m, l_n}, \nu_{k_m, l_n} \rangle \end{array} .
 \end{aligned}$$

$$\begin{aligned}
 & (\perp, \mathcal{O}_{\varphi_2, \psi_2}^2, \perp)A \\
 &= [K^*, \mathcal{O}_{\varphi_2, \psi_2}^2(L^*), \{\langle \mu_{k_i, l_j}, \nu_{k_i, l_j} \rangle\}] \\
 &= \begin{array}{c|ccc} & l_1, \mathcal{O}_{\varphi_2, \psi_2}^2(\langle \alpha_1^l, \beta_1^l \rangle) & \dots & l_n, \mathcal{O}_{\varphi_2, \psi_2}^2(\langle \alpha_n^l, \beta_n^l \rangle) \\ \hline k_1, \langle \alpha_1^k, \beta_1^k \rangle & \langle \mu_{k_1, l_1}, \nu_{k_1, l_1} \rangle & \dots & \langle \mu_{k_1, l_n}, \nu_{k_1, l_n} \rangle \\ \vdots & \vdots & \ddots & \vdots \\ k_i, \langle \alpha_i^k, \beta_i^k \rangle & \langle \mu_{k_i, l_1}, \nu_{k_i, l_1} \rangle & \dots & \langle \mu_{k_i, l_n}, \nu_{k_i, l_n} \rangle \\ \vdots & \vdots & \ddots & \vdots \\ k_m, \langle \alpha_m^k, \beta_m^k \rangle & \langle \mu_{k_m, l_1}, \nu_{k_m, l_1} \rangle & \dots & \langle \mu_{k_m, l_n}, \nu_{k_m, l_n} \rangle \end{array} .
 \end{aligned}$$

$$\begin{aligned}
 & (\perp, \perp, \mathcal{O}_{\varphi_3, \psi_3}^3)A \\
 &= [K^*, L^*, \{\mathcal{O}_{\varphi_3, \psi_3}^3(\langle \mu_{k_i, l_j}, \nu_{k_i, l_j} \rangle)\}] \\
 &= \begin{array}{c|ccc} & l_1, \langle \alpha_1^l, \beta_1^l \rangle & \dots & l_n, \langle \alpha_n^l, \beta_n^l \rangle \\ \hline k_1, \langle \alpha_1^k, \beta_1^k \rangle & \mathcal{O}_{\varphi_3, \psi_3}^3(\langle \mu_{k_1, l_1}, \nu_{k_1, l_1} \rangle) & \dots & \mathcal{O}_{\varphi_3, \psi_3}^3(\langle \mu_{k_1, l_n}, \nu_{k_1, l_n} \rangle) \\ \vdots & \vdots & \ddots & \vdots \\ k_i, \langle \alpha_i^k, \beta_i^k \rangle & \mathcal{O}_{\varphi_3, \psi_3}^3(\langle \mu_{k_i, l_1}, \nu_{k_i, l_1} \rangle) & \dots & \mathcal{O}_{\varphi_3, \psi_3}^3(\langle \mu_{k_i, l_n}, \nu_{k_i, l_n} \rangle) \\ \vdots & \vdots & \ddots & \vdots \\ k_m, \langle \alpha_m^k, \beta_m^k \rangle & \mathcal{O}_{\varphi_3, \psi_3}^3(\langle \mu_{k_m, l_1}, \nu_{k_m, l_1} \rangle) & \dots & \mathcal{O}_{\varphi_3, \psi_3}^3(\langle \mu_{k_m, l_n}, \nu_{k_m, l_n} \rangle) \end{array} .
 \end{aligned}$$

$$\begin{aligned}
& (\mathcal{O}_{\varphi_1, \psi_1}^1, \mathcal{O}_{\varphi_2, \psi_2}^2, \mathcal{O}_{\varphi_3, \psi_3}^3)A \\
&= (\mathcal{O}_{\varphi_1, \psi_1}^1, \perp, \perp)((\perp, \mathcal{O}_{\varphi_1, \psi_1}^1, \perp)((\perp, \perp, \mathcal{O}_{\varphi_1, \psi_1}^1)A)) \\
&= [\mathcal{O}_{\varphi_1, \psi_1}^1(K^*), \mathcal{O}_{\varphi_2, \psi_2}^2(L^*), \{\mathcal{O}_{\varphi_3, \psi_3}^3(\langle \mu_{k_i, l_j}, \nu_{k_i, l_j} \rangle)\}] \\
&= \begin{array}{c|ccc} & l_1, \mathcal{O}_{\varphi_2, \psi_2}^2(\langle \alpha_1^l, \beta_1^l \rangle) & \dots & l_n, \mathcal{O}_{\varphi_2, \psi_2}^2(\langle \alpha_n^l, \beta_n^l \rangle) \\ \hline k_1, \mathcal{O}_{\varphi_1, \psi_1}^1(\langle \alpha_1^k, \beta_1^k \rangle) & \mathcal{O}_{\varphi_3, \psi_3}^3(\langle \mu_{k_1, l_1}, \nu_{k_1, l_1} \rangle) & \dots & \mathcal{O}_{\varphi_3, \psi_3}^3(\langle \mu_{k_1, l_n}, \nu_{k_1, l_n} \rangle) \\ \vdots & \vdots & \ddots & \vdots \\ k_i, \mathcal{O}_{\varphi_1, \psi_1}^1(\langle \alpha_i^k, \beta_i^k \rangle) & \mathcal{O}_{\varphi_3, \psi_3}^3(\langle \mu_{k_i, l_1}, \nu_{k_i, l_1} \rangle) & \dots & \mathcal{O}_{\varphi_3, \psi_3}^3(\langle \mu_{k_i, l_n}, \nu_{k_i, l_n} \rangle) \\ \vdots & \vdots & \ddots & \vdots \\ k_m, \mathcal{O}_{\varphi_1, \psi_1}^1(\langle \alpha_m^k, \beta_m^k \rangle) & \mathcal{O}_{\varphi_3, \psi_3}^3(\langle \mu_{k_m, l_1}, \nu_{k_m, l_1} \rangle) & \dots & \mathcal{O}_{\varphi_3, \psi_3}^3(\langle \mu_{k_m, l_n}, \nu_{k_m, l_n} \rangle) \end{array},
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{O}_{\varphi_1, \psi_1}^1(K^*) &= \{\mathcal{O}_{\varphi_1, \psi_1}^1(\langle k_i, \alpha_i^K, \beta_i^K \rangle) | k_i \in K\} \\
&= \{\mathcal{O}_{\varphi_1, \psi_1}^1(\langle k_i, \alpha_i^K, \beta_i^K \rangle) | 1 \leq i \leq m\}, \\
\mathcal{O}_{\varphi_2, \psi_2}^2(L^*) &= \{\mathcal{O}_{\varphi_2, \psi_2}^2(\langle l_j, \alpha_j^L, \beta_j^L \rangle) | l_j \in L\} \\
&= \{\mathcal{O}_{\varphi_2, \psi_2}^2(\langle l_j, \alpha_j^L, \beta_j^L \rangle) | 1 \leq j \leq n\}.
\end{aligned}$$

For example,

$$\begin{aligned}
& (\square, F_{\alpha, \beta}, Q_{\alpha, \beta})A = [\square K^*, F_{\alpha, \beta}(L^*), \{\mathcal{O}_3(\langle \mu_{k_i, l_j}, \nu_{k_i, l_j} \rangle)\}] \\
&= [\{\square \langle k_i, \alpha_i^K, \beta_i^K \rangle | k_i \in K\}, \{F_{\alpha, \beta}(\langle l_j, \alpha_j^L, \beta_j^L \rangle) | l_j \in L\}, \\
&\quad \{Q_{\alpha, \beta}(\langle \mu_{k_i, l_j}, \nu_{k_i, l_j} \rangle)\}] \\
&= [\{\langle k_i, \alpha_i^K, 1 - \alpha_i^K \rangle | k_i \in K\}, \\
&\quad \{\langle l_j, \alpha_j^L + \alpha(1 - \alpha_j^L - \beta_j^L), \beta_j^L + \beta(1 - \alpha_j^L - \beta_j^L) \rangle | l_j \in L\}, \\
&\quad \{\langle \min(\mu_{k_i, l_j}, \alpha), \max(\nu_{k_i, l_j}, \beta) \rangle\}].
\end{aligned}$$

Operator $N_{\alpha, \beta}$ (see, e.g., [20]) gives different results. We will show them step-by-step. Let $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in [0, 1]$ and $\alpha + \beta \leq 1, \gamma + \delta \leq 1, \varepsilon + \zeta \leq 1$. Then

$$\begin{aligned}
(N_{\alpha, \beta}, \perp, \perp)A &= [N_{\alpha, \beta}K^*, L^*, \{\langle \mu_{k_i, l_j}, \nu_{k_i, l_j} \rangle\}] \\
&= [U^*, L^*, \{\langle \mu_{k_i, l_j}, \nu_{k_i, l_j} \rangle\}], \\
(\perp, N_{\gamma, \delta}, \perp)A &= [K^*, N_{\gamma, \delta}L^*, \{\langle \mu_{k_i, l_j}, \nu_{k_i, l_j} \rangle\}] \\
&= [K^*, V^*, \{\langle \mu_{k_i, l_j}, \nu_{k_i, l_j} \rangle\}], \\
(\perp, \perp, N_{\varepsilon, \zeta})A &= @[K^*, L^*, N_{\varepsilon, \zeta}\{\langle \mu_{k_i, l_j}, \nu_{k_i, l_j} \rangle\}] \\
&= @[K^*, L^*, \{N_{\varepsilon, \zeta}(\mu_{k_i, l_j}, \nu_{k_i, l_j})\}] \\
&= @[K^*, L^*, \{\langle \mu_{k_i, l_j}, \nu_{k_i, l_j} \rangle \mid \mu_{k_i, l_j} \geq \varepsilon, \nu_{k_i, l_j} \leq \zeta\}], \\
&= [R^*, S^*, \{\langle \rho_{k_i, l_j}, \sigma_{k_i, l_j} \rangle\}],
\end{aligned}$$

where:

$$U^* = \{\langle k_i, \alpha_i^k, \beta_i^k \rangle \mid \alpha_i^k \geq \alpha \ \& \ \beta_i^k \leq \beta \ \& \ x \in E\} \subseteq K^*,$$

$$V^* = \{\langle l_j, \alpha_j^l, \beta_j^l \rangle \mid \alpha_j^l \geq \gamma \ \& \ \beta_j^l \leq \delta \ \& \ x \in E\} \subseteq L^*,$$

and the EIFIM $[R^*, S^*, \{\langle \rho_{k_i, l_j}, \sigma_{k_i, l_j} \rangle\}]$ is obtained by the procedure described in Section 2.3.

When $A = [K, L, \{\langle \mu_{k_i, l_j}, \nu_{k_i, l_j} \rangle\}]$ is an IFIM, then the triple of operators is modified to $(\perp, \perp, \mathcal{O}_{\varphi, \psi}^3)A$ or (for brevity) restricted to

$$\mathcal{O}_{\varphi, \psi} A = [K, L, \{\mathcal{O}_{\varphi, \psi}(\langle \mu_{k_i, l_j}, \nu_{k_i, l_j} \rangle)\}]$$

$$= \begin{array}{c|ccc} & l_1 & \dots & l_n \\ \hline k_1 & O_{\varphi, \psi}(\langle \mu_{k_1, l_1}, \nu_{k_1, l_1} \rangle) & \dots & O_{\varphi, \psi}(\langle \mu_{k_1, l_n}, \nu_{k_1, l_n} \rangle) \\ \vdots & \vdots & \ddots & \vdots \\ k_i & O_{\varphi, \psi}(\langle \mu_{k_i, l_1}, \nu_{k_i, l_1} \rangle) & \dots & O_{\varphi, \psi}(\langle \mu_{k_i, l_n}, \nu_{k_i, l_n} \rangle) \\ \vdots & \vdots & \ddots & \vdots \\ k_m & O_{\varphi, \psi}(\langle \mu_{k_m, l_1}, \nu_{k_m, l_1} \rangle) & \dots & O_{\varphi, \psi}(\langle \mu_{k_m, l_n}, \nu_{k_m, l_n} \rangle) \end{array}.$$

4.6 Examples from graph theory. Part 3

Let $V = \{v_1, v_2, \dots, v_n\}$ be a fixed set of vertices and let each vertex x have a degree of existence $\alpha(x)$ and a degree of non-existence $\beta(x)$. Therefore, we can construct the IFS

$$V^* = \{\langle v, \alpha(v), \beta(v) \rangle \mid v \in V\} = \{\langle v_i, \alpha(v_i), \beta(v_i) \rangle \mid 1 \leq i \leq n\},$$

where for each $v \in V$: $\alpha(v), \beta(v), \alpha(v) + \beta(v) \in [0, 1]$.

Let H be a set of arcs between the vertices from V . Following [17], we can again juxtapose to each arc a degree of existence $\mu(x, y)$ and a degree of non-existence $\nu(x, y)$. Therefore, we can construct the new IFS

$$H^* = \{\langle \langle x, y \rangle, \mu(x, y), \nu(x, y) \rangle \mid x, y \in V\}$$

$$= \{\langle \langle v_i, v_j \rangle, \mu(v_i, v_j), \nu(v_i, v_j) \rangle \mid 1 \leq i, j \leq n\},$$

where for each $x, y \in V$:

$$\mu(x, y), \nu(x, y), \mu(x, y) + \nu(x, y) \in [0, 1].$$

Now, for the graph $G = (V, H)$ we can construct the Extended Intuitionistic Fuzzy Graph (EIFG) $G^* = (V^*, H^*)$. It has the following IM-representation:

$$[V^*, V^*, \{\langle \mu(v_i, v_j), \nu(v_i, v_j) \rangle\}]$$

$$= \begin{array}{c|ccc} & v_1, \langle \alpha(v_1), \beta(v_1) \rangle & \cdots & v_n, \langle \alpha(v_n), \beta(v_n) \rangle \\ \hline v_1, \langle \alpha(v_1), \beta(v_1) \rangle & \langle \mu_{v_1, v_1}, \nu_{v_1, v_1} \rangle & \cdots & \langle \mu_{v_1, v_n}, \nu_{v_1, v_n} \rangle \\ \vdots & \vdots & \ddots & \vdots \\ v_i, \langle \alpha(v_i), \beta(v_i) \rangle & \langle \mu_{v_i, v_1}, \nu_{v_i, v_1} \rangle & \cdots & \langle \mu_{v_i, v_n}, \nu_{v_i, v_n} \rangle \\ \vdots & \vdots & \ddots & \vdots \\ v_n, \langle \alpha(v_n), \beta(v_n) \rangle & \langle \mu_{v_n, v_1}, \nu_{v_n, v_1} \rangle & \cdots & \langle \mu_{v_n, v_n}, \nu_{v_n, v_n} \rangle \end{array},$$

where for every $1 \leq i \leq n, 1 \leq j \leq n$: $\mu_{v_i, v_j}, \nu_{v_i, v_j} \in [0, 1], \mu_{v_i, v_j} + \nu_{v_i, v_j} \in [0, 1], \alpha(v_i), \beta(v_i) \in [0, 1], \alpha(v_i) + \beta(v_i) \in [0, 1]$.

Let us discuss here for simplicity only the case of an oriented graph. Let us denote by $x \rightarrow y$ the fact that both vertices x and y are connected by an arc and x is the source, while y is the target. Let operation $\circ \in \{+, \max, @, \min, \times\}$.

We call that the EIFG G^* is “well-top-down-(very strong, strong, middle, weak, very weak)-ordered”, or shortly, “well-top-down-o-ordered”, if for every two vertices v_i and v_j , such that $v_i \rightarrow v_j$, the following inequality holds:

$$\langle \alpha_i, \beta_i \rangle \circ \langle \mu_{v_i, v_j}, \nu_{v_i, v_j} \rangle \geq \langle \alpha_j, \beta_j \rangle.$$

Analogously, we call that the EIFG G^* is “well-bottom-up-(very strong, strong, middle, weak, very weak)-ordered”, or shortly, “well-bottom-up-o-ordered”, if for every two vertices v_i and v_j , such that $v_i \rightarrow v_j$, the following inequality holds:

$$\langle \alpha_i, \beta_i \rangle \circ \langle \mu_{v_i, v_j}, \nu_{v_i, v_j} \rangle \leq \langle \alpha_j, \beta_j \rangle.$$

We illustrate the IM-interpretation of the EIFGs by the following example. Let us have the EIFG G^* of the form Figure 4.1.

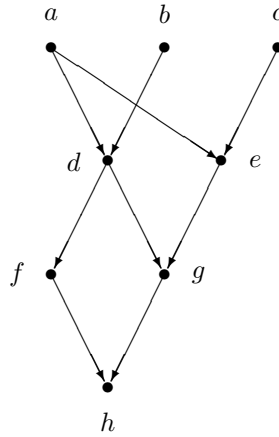


Figure 4.1. EIFG G^*

The H^* -component and the arcs of Hj with their evaluations have the following form:

$$H^* = [\{\langle a, \frac{1}{2}, \frac{1}{3} \rangle, \langle b, \frac{1}{3}, \frac{1}{3} \rangle, \langle c, \frac{1}{3}, \frac{1}{2} \rangle, \langle d, \frac{2}{3}, \frac{1}{8} \rangle, \langle e, \frac{3}{4}, \frac{1}{4} \rangle, \langle f, \frac{1}{10}, \frac{7}{8} \rangle, \langle g, \frac{2}{5}, \frac{3}{5} \rangle, \langle h, \frac{1}{5}, \frac{1}{5} \rangle\}, \\ \{\langle a, \frac{1}{2}, \frac{1}{3} \rangle, \langle b, \frac{1}{3}, \frac{1}{3} \rangle, \langle c, \frac{1}{3}, \frac{1}{2} \rangle, \langle d, \frac{2}{3}, \frac{1}{8} \rangle, \langle e, \frac{3}{4}, \frac{1}{4} \rangle, \langle f, \frac{1}{10}, \frac{7}{8} \rangle, \langle g, \frac{2}{5}, \frac{3}{5} \rangle, \langle h, \frac{1}{5}, \frac{1}{5} \rangle\}, \\ \{\mu_{x,y}, \nu_{x,y}\}].$$

Now, having in mind the discussion in Section 2.7, we can modify the IM to the form

$$H^* = [\{\langle a, \frac{1}{2}, \frac{1}{3} \rangle, \langle b, \frac{1}{3}, \frac{1}{3} \rangle, \langle c, \frac{1}{3}, \frac{1}{2} \rangle, \langle d, \frac{2}{3}, \frac{1}{8} \rangle, \langle e, \frac{3}{4}, \frac{1}{4} \rangle, \langle f, \frac{1}{10}, \frac{7}{8} \rangle, \langle g, \frac{2}{5}, \frac{3}{5} \rangle\}, \\ \{\langle d, \frac{2}{3}, \frac{1}{8} \rangle, \langle e, \frac{3}{4}, \frac{1}{4} \rangle, \langle f, \frac{1}{10}, \frac{7}{8} \rangle, \langle g, \frac{2}{5}, \frac{3}{5} \rangle, \langle h, \frac{1}{5}, \frac{1}{5} \rangle\}, \\ \{\mu_{x,y}, \nu_{x,y}\}].$$

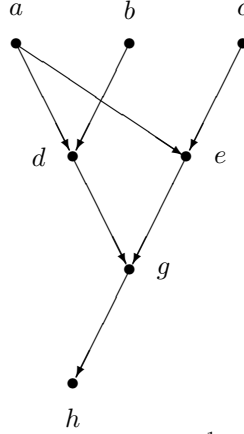
The form of the new IM is

	$d, \langle \frac{2}{3}, \frac{1}{8} \rangle$	$e, \langle \frac{3}{4}, \frac{1}{4} \rangle$	$f, \langle \frac{1}{10}, \frac{7}{8} \rangle$	$g, \langle \frac{2}{5}, \frac{3}{5} \rangle$	$h, \langle \frac{1}{5}, \frac{1}{5} \rangle$
$a, \langle \frac{1}{2}, \frac{1}{3} \rangle$	$\langle \frac{3}{4}, \frac{1}{5} \rangle$	$\langle \frac{1}{2}, \frac{1}{4} \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$
$b, \langle \frac{1}{3}, \frac{1}{3} \rangle$	$\langle \frac{2}{3}, 0 \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$
$c, \langle \frac{1}{3}, \frac{1}{2} \rangle$	$\langle 0, 1 \rangle$	$\langle \frac{1}{5}, \frac{2}{5} \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$
$d, \langle \frac{2}{3}, \frac{1}{8} \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$	$\langle \frac{1}{5}, \frac{2}{5} \rangle$	$\langle \frac{3}{4}, \frac{1}{5} \rangle$	$\langle 0, 1 \rangle$
$e, \langle \frac{3}{4}, \frac{1}{4} \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$	$\langle \frac{2}{3}, \frac{1}{6} \rangle$	$\langle 0, 1 \rangle$
$f, \langle \frac{1}{10}, \frac{7}{8} \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$	$\langle \frac{1}{3}, \frac{1}{4} \rangle$
$g, \langle \frac{2}{5}, \frac{3}{5} \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$	$\langle \frac{3}{5}, \frac{1}{5} \rangle$

Now, we can apply one or more of the level-operators $N_{\rho_1, \sigma_1}^1, N_{\rho_2, \sigma_2}^2, N_{\rho_3, \sigma_3}^3$ and as a result, the form of the graph will be changed. It is important to mention that in the present case (when the two index sets coincide), the first two level operators must have equal parameters and, therefore, if some vertex has to be omitted from one of both index sets, it will be omitted from the other index set, too. For example, if we apply operator $N_{\frac{1}{5}, \frac{1}{4}}^1$ over G^* , we obtain

	$d, \langle \frac{2}{3}, \frac{1}{8} \rangle$	$e, \langle \frac{3}{4}, \frac{1}{4} \rangle$	$g, \langle \frac{2}{5}, \frac{3}{5} \rangle$	$h, \langle \frac{1}{5}, \frac{1}{5} \rangle$
$a, \langle \frac{1}{2}, \frac{1}{3} \rangle$	$\langle \frac{3}{4}, \frac{1}{5} \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$
$b, \langle \frac{1}{3}, \frac{1}{3} \rangle$	$\langle \frac{2}{3}, 0 \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$
$c, \langle \frac{1}{3}, \frac{1}{2} \rangle$	$\langle 0, 1 \rangle$	$\langle \frac{1}{5}, \frac{2}{5} \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$
$d, \langle \frac{2}{3}, \frac{1}{8} \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$	$\langle \frac{3}{4}, \frac{1}{5} \rangle$	$\langle 0, 1 \rangle$
$e, \langle \frac{3}{4}, \frac{1}{4} \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$	$\langle \frac{2}{3}, \frac{1}{6} \rangle$	$\langle 0, 1 \rangle$
$g, \langle \frac{2}{5}, \frac{3}{5} \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$	$\langle \frac{3}{5}, \frac{1}{5} \rangle$

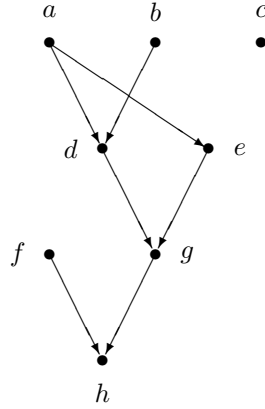
and the new graph has the form Figure 4.2.


 Figure 4.2. An EIFG $N_{\frac{1}{5}, \frac{1}{4}}^1(G^*)$

On the other hand, if we can apply, e.g., operator $N_{\frac{1}{4}, \frac{1}{3}}^3$ over G^* , we obtain

	$d, \langle \frac{2}{3}, \frac{1}{8} \rangle$	$e, \langle \frac{3}{4}, \frac{1}{4} \rangle$	$f, \langle \frac{1}{10}, \frac{7}{8} \rangle$	$g, \langle \frac{2}{5}, \frac{3}{5} \rangle$	$h, \langle \frac{1}{5}, \frac{1}{5} \rangle$
$a, \langle \frac{1}{2}, \frac{1}{3} \rangle$	$\langle \frac{3}{4}, \frac{1}{5} \rangle$	$\langle \frac{1}{2}, \frac{1}{4} \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$
$b, \langle \frac{1}{3}, \frac{1}{3} \rangle$	$\langle \frac{2}{3}, 0 \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$
$c, \langle \frac{1}{3}, \frac{1}{2} \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$
$d, \langle \frac{2}{3}, \frac{1}{8} \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$	$\langle \frac{3}{4}, \frac{1}{5} \rangle$	$\langle 0, 1 \rangle$
$e, \langle \frac{3}{4}, \frac{1}{4} \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$	$\langle \frac{2}{3}, \frac{1}{6} \rangle$	$\langle 0, 1 \rangle$
$f, \langle \frac{1}{10}, \frac{7}{8} \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$	$\langle \frac{1}{3}, \frac{1}{4} \rangle$
$g, \langle \frac{2}{5}, \frac{3}{5} \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$	$\langle \frac{3}{5}, \frac{1}{5} \rangle$

and the new graph has the form Figure 4.3.


 Figure 4.3. An EIFG $N_{\frac{1}{4}, \frac{1}{3}}^3$

and the IM G^* can be reduced to

	$d, \langle \frac{2}{3}, \frac{1}{8} \rangle$	$e, \langle \frac{3}{4}, \frac{1}{4} \rangle$	$g, \langle \frac{2}{5}, \frac{3}{5} \rangle$	$h, \langle \frac{1}{5}, \frac{1}{5} \rangle$
$a, \langle \frac{1}{2}, \frac{1}{3} \rangle$	$\langle \frac{3}{4}, \frac{1}{5} \rangle$	$\langle \frac{1}{2}, \frac{1}{4} \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$
$b, \langle \frac{1}{3}, \frac{1}{3} \rangle$	$\langle \frac{2}{3}, 0 \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$
$c, \langle \frac{1}{3}, \frac{1}{2} \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$
$d, \langle \frac{2}{3}, \frac{1}{8} \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$	$\langle \frac{3}{4}, \frac{1}{5} \rangle$	$\langle 0, 1 \rangle$
$e, \langle \frac{3}{4}, \frac{1}{4} \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$	$\langle \frac{2}{3}, \frac{1}{6} \rangle$	$\langle 0, 1 \rangle$
$f, \langle \frac{1}{10}, \frac{7}{8} \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$	$\langle \frac{1}{3}, \frac{1}{4} \rangle$
$g, \langle \frac{2}{5}, \frac{3}{5} \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$	$\langle \frac{3}{5}, \frac{1}{5} \rangle$

Obviously, vertex f remains in the first index set, because an arc goes out of it. On the other hand, vertex c here is an isolated one.

The present text was published in [17] and related results were published in [67, 69].

Let us have the following two simple graphs X and Y Figure 4.4:

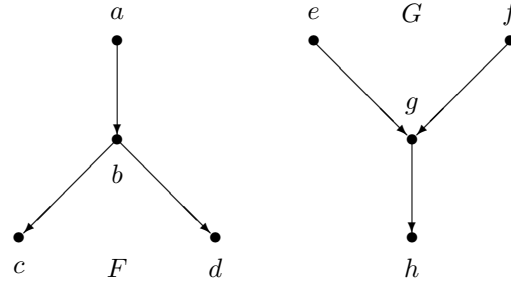


Figure 4.4. Graphs X and Y

and let their respective EIFIMs be

$$X^* = \begin{array}{c|ccc} & b, \langle \gamma_1, \delta_1 \rangle & c, \langle \gamma_2, \delta_2 \rangle & d, \langle \gamma_3, \delta_3 \rangle \\ \hline a, \langle \alpha_1, \beta_1 \rangle & \langle \mu_{a,b}, \nu_{a,b} \rangle & \langle 0, 1 \rangle & \langle 0, 1 \rangle \\ b, \langle \alpha_2, \beta_2 \rangle & \langle 0, 1 \rangle & \langle \mu_{b,c}, \nu_{b,c} \rangle & \langle \mu_{b,d}, \nu_{b,d} \rangle \end{array}$$

and

$$Y^* = \begin{array}{c|cc} & g, \langle \eta_1, \theta_1 \rangle & h, \langle \eta_2, \theta_2 \rangle \\ \hline e, \langle \varepsilon_1, \zeta_1 \rangle & \langle \mu_{e,g}, \nu_{e,g} \rangle & \langle 0, 1 \rangle \\ f, \langle \varepsilon_2, \zeta_2 \rangle & \langle \mu_{f,g}, \nu_{f,g} \rangle & \langle 0, 1 \rangle \\ g, \langle \varepsilon_3, \zeta_3 \rangle & \langle 0, 1 \rangle & \langle \mu_{g,h}, \nu_{g,h} \rangle \end{array}.$$

If we know that vertices a and e ; b and g ; and d and h coincide, then by operator “substitution” defined over IM from EIFIM Y^* we obtain the new EIFIM

$$Z^* = \begin{bmatrix} e & g & h \\ a & b & d \end{bmatrix} Y^* = \begin{array}{c|cc} & b, \langle \eta_1, \theta_1 \rangle & d, \langle \eta_2, \theta_2 \rangle \\ \hline a, \langle \varepsilon_1, \zeta_1 \rangle & \langle \mu_{e,g}, \nu_{e,g} \rangle & \langle 0, 1 \rangle \\ f, \langle \varepsilon_2, \zeta_2 \rangle & \langle \mu_{f,g}, \nu_{f,g} \rangle & \langle 0, 1 \rangle \\ b, \langle \varepsilon_3, \zeta_3 \rangle & \langle 0, 1 \rangle & \langle \mu_{g,h}, \nu_{g,h} \rangle \end{array}.$$

We must mention that we keep deliberately the old indices of μ and ν function, because in the opposite case on the next step there may be a confusion.

Now, if we would like to unite IFGs X and Z using their EIFIM-representations X^* and Z^* , then we will obtain the graph Figure 4.5:

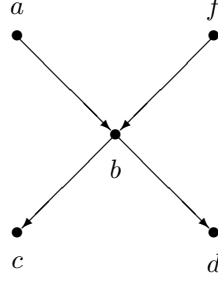


Figure 4.5. The union of graphs X and Z

which has the EIFIM-representation

$$X^* \oplus_{(*,*,\circ)} Z^* = \begin{array}{c|ccc} & b, \langle \gamma_1, \delta_1 \rangle * \langle \eta_1, \theta_1 \rangle & c, \langle \gamma_2, \delta_2 \rangle & d, \langle \gamma_3, \delta_3 \rangle * \langle \eta_2, \theta_2 \rangle \\ \hline a, \langle \alpha_1, \beta_1 \rangle * \langle \varepsilon_1, \zeta_1 \rangle & \langle \mu_{a,b}, \nu_{a,b} \rangle \circ \langle \varepsilon_1, \zeta_1 \rangle & \langle 0, 1 \rangle & \langle 0, 1 \rangle \\ b, \langle \alpha_2, \beta_2 \rangle * \langle \varepsilon_3, \zeta_3 \rangle & \langle 0, 1 \rangle & \langle \mu_{b,c}, \nu_{b,c} \rangle & \langle \mu_{b,d}, \nu_{b,d} \rangle \circ \langle \mu_{g,h}, \nu_{g,h} \rangle \\ f, \langle \varepsilon_2, \zeta_2 \rangle & \langle \mu_{f,g}, \nu_{f,g} \rangle & \langle 0, 1 \rangle & \langle 0, 1 \rangle \end{array}$$

What will be the sense of operations $\circ$ and $*$ depends on the particular situation. For example, we may want to decrease the degree of validity (existence) and increase the degree of non-validity (non-existence) of the vertices of the new graph and simultaneously to increase the degree of validity (existence) and decrease the degree of non-validity (non-existence) of the arcs. Then operation $\circ$ shall be $\vee$ and operation $*$ shall be $\wedge$. Obviously, this situation cannot be obtained with the standard form of the operation $\oplus$.

Similarly, we can illustrate the two other operations $\otimes$ and $\odot$.

4.7 Level operators over IFIMs and EIFIMs

As it is mentioned in [53], there are two groups of level operators. The operators from the first group are applicable as over EIFIMs, as well as over IFIMs, while the operators from the second group are specific only for EIFIMs.

Let us have the EIFIM $A = [K^*, L^*, \langle \mu_{k_i, l_j}, \nu_{k_i, l_j} \rangle]$, where $\langle \mu_{k_i, l_j}, \nu_{k_i, l_j} \rangle$ is an IFP. Then we define for two fixed numbers $\alpha, \beta \in [0, 1]$ such that $\alpha + \beta \leq 1$:

$$\mathcal{L}_{\alpha, \beta}^>(A) = [K^*, L^*, \{\langle \rho_{k_i, l_j}, \sigma_{k_i, l_j} \rangle\}],$$

$$\text{where } \langle \rho_{k_i, l_j}, \sigma_{k_i, l_j} \rangle = \begin{cases} \langle \mu_{k_i, l_j}, \nu_{k_i, l_j} \rangle, & \text{if } \langle \mu_{k_i, l_j}, \nu_{k_i, l_j} \rangle > \langle \alpha, \beta \rangle \\ \langle 0, 1 \rangle, & \text{otherwise} \end{cases};$$

$$\mathcal{L}_{\alpha, \beta}^{\geq}(A) = [K^*, L^*, \{\langle \rho_{k_i, l_j}, \sigma_{k_i, l_j} \rangle\}],$$

$$\text{where } \langle \rho_{k_i, l_j}, \sigma_{k_i, l_j} \rangle = \begin{cases} \langle \mu_{k_i, l_j}, \nu_{k_i, l_j} \rangle, & \text{if } \langle \mu_{k_i, l_j}, \nu_{k_i, l_j} \rangle \geq \langle \alpha, \beta \rangle \\ \langle 0, 1 \rangle, & \text{otherwise} \end{cases};$$

$$\mathcal{L}_{\alpha, \beta}^<(A) = [K^*, L^*, \{\langle \rho_{k_i, l_j}, \sigma_{k_i, l_j} \rangle\}],$$

$$\text{where } \langle \rho_{k_i, l_j}, \sigma_{k_i, l_j} \rangle = \begin{cases} \langle \mu_{k_i, l_j}, \nu_{k_i, l_j} \rangle, & \text{if } \langle \mu_{k_i, l_j}, \nu_{k_i, l_j} \rangle < \langle \alpha, \beta \rangle \\ \langle 1, 0 \rangle, & \text{otherwise} \end{cases};$$

$$\mathcal{L}_{\alpha, \beta}^{\leq}(A) = [K^*, L^*, \{\langle \rho_{k_i, l_j}, \sigma_{k_i, l_j} \rangle\}],$$

$$\text{where } \langle \rho_{k_i, l_j}, \sigma_{k_i, l_j} \rangle = \begin{cases} \langle \mu_{k_i, l_j}, \nu_{k_i, l_j} \rangle, & \text{if } \langle \mu_{k_i, l_j}, \nu_{k_i, l_j} \rangle \leq \langle \alpha, \beta \rangle \\ \langle 1, 0 \rangle, & \text{otherwise} \end{cases}.$$

Let for $K, L \subset \mathcal{I}$:

$$O_{K^*, L^*}^* = [K^*, L^*, \{\langle 0, 1 \rangle\}],$$

$$E_{K^*, L^*}^* = [K^*, L^*, \{\langle 1, 0 \rangle\}]$$

denote the zero and unit EIFIMs with index sets K^* and L^* . Obviously, the following equalities are valid for every two fixed numbers $\alpha, \beta \in [0, 1]$ such that $\alpha + \beta \leq 1$:

$$\mathcal{L}_{\alpha, \beta}^>(O_{K^*, L^*}^*) = O_{K^*, L^*}^*,$$

$$\mathcal{L}_{\alpha, \beta}^{\geq}(O_{K^*, L^*}^*) = O_{K^*, L^*}^*,$$

$$\mathcal{L}_{\alpha, \beta}^<(O_{K^*, L^*}^*) = \begin{cases} O_{K^*, L^*}^*, & \text{if } \langle \alpha, \beta \rangle > \langle 0, 1 \rangle \\ E_{K^*, L^*}^*, & \text{if } \langle \alpha, \beta \rangle = \langle 0, 1 \rangle \end{cases},$$

$$\mathcal{L}_{\alpha, \beta}^{\leq}(O_{K^*, L^*}^*) = O_{K^*, L^*}^*,$$

$$\mathcal{L}_{\alpha, \beta}^>(E_{K^*, L^*}^*) = \begin{cases} E_{K^*, L^*}^*, & \text{if } \langle \alpha, \beta \rangle > \langle 0, 1 \rangle \\ O_{K^*, L^*}^*, & \text{if } \langle \alpha, \beta \rangle = \langle 0, 1 \rangle \end{cases},$$

$$\mathcal{L}_{\alpha, \beta}^{\geq}(E_{K^*, L^*}^*) = E_{K^*, L^*}^*,$$

$$\mathcal{L}_{\alpha, \beta}^<(E_{K^*, L^*}^*) = E_{K^*, L^*}^*,$$

$$\mathcal{L}_{\alpha, \beta}^{\leq}(E_{K^*, L^*}^*) = E_{K^*, L^*}^*.$$

In [53], it is proved that for each EIFIM A and for every two IFPs $\langle \alpha, \beta \rangle, \langle \gamma, \delta \rangle$:

$$\begin{aligned}\mathcal{L}_{\langle \alpha, \beta \rangle}^>(\mathcal{L}_{\langle \gamma, \delta \rangle}^>(A)) &= \mathcal{L}_{\langle \max(\alpha, \gamma), \min(\beta, \delta) \rangle}^>(A), \\ \mathcal{L}_{\langle \alpha, \beta \rangle}^{\geq}(\mathcal{L}_{\langle \gamma, \delta \rangle}^{\geq}(A)) &= \mathcal{L}_{\langle \max(\alpha, \gamma), \min(\beta, \delta) \rangle}^{\geq}(A), \\ \mathcal{L}_{\langle \alpha, \beta \rangle}^<(\mathcal{L}_{\langle \gamma, \delta \rangle}^<(A)) &= \mathcal{L}_{\langle \min(\alpha, \gamma), \max(\beta, \delta) \rangle}^<(A), \\ \mathcal{L}_{\langle \alpha, \beta \rangle}^{\leq}(\mathcal{L}_{\langle \gamma, \delta \rangle}^{\leq}(A)) &= \mathcal{L}_{\langle \min(\alpha, \gamma), \max(\beta, \delta) \rangle}^{\leq}(A);\end{aligned}$$

and also

$$\begin{aligned}\mathcal{L}_{\langle \alpha, \beta \rangle}^>(\mathcal{L}_{\langle \gamma, \delta \rangle}^>(A)) &= \mathcal{L}_{\langle \gamma, \delta \rangle}^>(\mathcal{L}_{\langle \alpha, \beta \rangle}^>(A)), \\ \mathcal{L}_{\langle \alpha, \beta \rangle}^{\geq}(\mathcal{L}_{\langle \gamma, \delta \rangle}^{\geq}(A)) &= \mathcal{L}_{\langle \gamma, \delta \rangle}^{\geq}(\mathcal{L}_{\langle \alpha, \beta \rangle}^{\geq}(A)), \\ \mathcal{L}_{\langle \alpha, \beta \rangle}^<(\mathcal{L}_{\langle \gamma, \delta \rangle}^<(A)) &= \mathcal{L}_{\langle \gamma, \delta \rangle}^<(\mathcal{L}_{\langle \alpha, \beta \rangle}^<(A)), \\ \mathcal{L}_{\langle \alpha, \beta \rangle}^{\leq}(\mathcal{L}_{\langle \gamma, \delta \rangle}^{\leq}(A)) &= \mathcal{L}_{\langle \gamma, \delta \rangle}^{\leq}(\mathcal{L}_{\langle \alpha, \beta \rangle}^{\leq}(A)).\end{aligned}$$

In the same paper it is proved that if the two EIFIMs A and B are given and if $\langle \alpha, \beta \rangle$ is an arbitrary IFP, then

- (a) $\mathcal{L}_{\langle \alpha, \beta \rangle}^>(A) \oplus_{(+)} \mathcal{L}_{\langle \alpha, \beta \rangle}^>(B) \subseteq_v \mathcal{L}_{\langle \alpha, \beta \rangle}^>(A \oplus_{(+)} B),$
- (b) $\mathcal{L}_{\langle \alpha, \beta \rangle}^>(A) \oplus_{(\cdot)} \mathcal{L}_{\langle \alpha, \beta \rangle}^>(B) \supseteq_v \mathcal{L}_{\langle \alpha, \beta \rangle}^>(A \oplus_{(\cdot)} B),$
- (c) $\mathcal{L}_{\langle \alpha, \beta \rangle}^>(A) \oplus_{(\vee)} \mathcal{L}_{\langle \alpha, \beta \rangle}^>(B) \subseteq_v \mathcal{L}_{\langle \alpha, \beta \rangle}^>(A \oplus_{(\vee)} B),$
- (d) $\mathcal{L}_{\langle \alpha, \beta \rangle}^>(A) \oplus_{(\wedge)} \mathcal{L}_{\langle \alpha, \beta \rangle}^>(B) \supseteq_v \mathcal{L}_{\langle \alpha, \beta \rangle}^>(A \oplus_{(\wedge)} B),$
- (e) $\mathcal{L}_{\langle \alpha, \beta \rangle}^{\geq}(A) \oplus_{(+)} \mathcal{L}_{\langle \alpha, \beta \rangle}^{\geq}(B) \subseteq_v \mathcal{L}_{\langle \alpha, \beta \rangle}^{\geq}(A \oplus_{(+)} B),$
- (f) $\mathcal{L}_{\langle \alpha, \beta \rangle}^{\geq}(A) \oplus_{(\cdot)} \mathcal{L}_{\langle \alpha, \beta \rangle}^{\geq}(B) \supseteq_v \mathcal{L}_{\langle \alpha, \beta \rangle}^{\geq}(A \oplus_{(\cdot)} B),$
- (g) $\mathcal{L}_{\langle \alpha, \beta \rangle}^{\geq}(A) \oplus_{(\vee)} \mathcal{L}_{\langle \alpha, \beta \rangle}^{\geq}(B) \subseteq_v \mathcal{L}_{\langle \alpha, \beta \rangle}^{\geq}(A \oplus_{(\vee)} B),$
- (h) $\mathcal{L}_{\langle \alpha, \beta \rangle}^{\geq}(A) \oplus_{(\wedge)} \mathcal{L}_{\langle \alpha, \beta \rangle}^{\geq}(B) \supseteq_v \mathcal{L}_{\langle \alpha, \beta \rangle}^{\geq}(A \oplus_{(\wedge)} B),$
- (i) $\mathcal{L}_{\langle \alpha, \beta \rangle}^<(A) \oplus_{(+)} \mathcal{L}_{\langle \alpha, \beta \rangle}^<(B) \subseteq_v \mathcal{L}_{\langle \alpha, \beta \rangle}^<(A \oplus_{(+)} B),$
- (j) $\mathcal{L}_{\langle \alpha, \beta \rangle}^<(A) \oplus_{(\cdot)} \mathcal{L}_{\langle \alpha, \beta \rangle}^<(B) \supseteq_v \mathcal{L}_{\langle \alpha, \beta \rangle}^<(A \oplus_{(\cdot)} B),$
- (k) $\mathcal{L}_{\langle \alpha, \beta \rangle}^<(A) \oplus_{(\vee)} \mathcal{L}_{\langle \alpha, \beta \rangle}^<(B) \subseteq_v \mathcal{L}_{\langle \alpha, \beta \rangle}^<(A \oplus_{(\vee)} B),$
- (l) $\mathcal{L}_{\langle \alpha, \beta \rangle}^<(A) \oplus_{(\wedge)} \mathcal{L}_{\langle \alpha, \beta \rangle}^<(B) \supseteq_v \mathcal{L}_{\langle \alpha, \beta \rangle}^<(A \oplus_{(\wedge)} B),$
- (m) $\mathcal{L}_{\langle \alpha, \beta \rangle}^{\leq}(A) \oplus_{(+)} \mathcal{L}_{\langle \alpha, \beta \rangle}^{\leq}(B) \subseteq_v \mathcal{L}_{\langle \alpha, \beta \rangle}^{\leq}(A \oplus_{(+)} B),$
- (n) $\mathcal{L}_{\langle \alpha, \beta \rangle}^{\leq}(A) \oplus_{(\cdot)} \mathcal{L}_{\langle \alpha, \beta \rangle}^{\leq}(B) \supseteq_v \mathcal{L}_{\langle \alpha, \beta \rangle}^{\leq}(A \oplus_{(\cdot)} B),$
- (o) $\mathcal{L}_{\langle \alpha, \beta \rangle}^{\leq}(A) \oplus_{(\vee)} \mathcal{L}_{\langle \alpha, \beta \rangle}^{\leq}(B) \subseteq_v \mathcal{L}_{\langle \alpha, \beta \rangle}^{\leq}(A \oplus_{(\vee)} B),$
- (p) $\mathcal{L}_{\langle \alpha, \beta \rangle}^{\leq}(A) \oplus_{(\wedge)} \mathcal{L}_{\langle \alpha, \beta \rangle}^{\leq}(B) \supseteq_v \mathcal{L}_{\langle \alpha, \beta \rangle}^{\leq}(A \oplus_{(\wedge)} B).$

As it is mentioned in [53], if we change operation $\oplus_{(\circ)}$ with operation $\otimes_{(\circ)}$ or $\odot_{(\circ, *)}$, the inequalities will keep their forms and the proofs will be analogous.

Now, we will modify the operator **automatic reduction** from [20] to the form:

$$@ (A) = [P^*, Q^*, \{b_{p_r, q_s}\}],$$

where $P \subseteq K, Q \subseteq L$ are index sets with the following property:

$$\begin{aligned} & (\forall k \in K^* - P^*)(\forall l \in L^*)(a_{k_i, l_j} = \langle 0, 1 \rangle) \ \& \ (\forall k \in K^*)(\forall l \in L^* - Q^*) \\ & (a_{k_i, l_j} = \langle 0, 1 \rangle) \ \& \ (\forall p_r = k_i \in P)(\forall q_s = l_j \in Q)(b_{p_r, q_s} = a_{k_i, l_j}). \end{aligned}$$

Therefore, this operator removes the rows and columns of EIFIM A that contain only elements $\langle 0, 1 \rangle$.

Now, we will give definitions of the operators from the second group and will discuss some of their properties.

Let us have the EIFIM $A = [K^*, L^*, \{\langle \mu_{k_i, l_j}, \nu_{k_i, l_j} \rangle\}]$ and for the two numbers $\alpha, \beta \in [0, 1]$ such that $\alpha + \beta \leq 1$:

$$\mathcal{L}_{R, \alpha, \beta}^>(A) = [\overline{K}^*, L^*, \{\langle \mu_{k_i, l_j}, \nu_{k_i, l_j} \rangle\}],$$

where here and below $\overline{K}^* = \{\langle k_i, \gamma_i^k, \delta_i^k \rangle | k_i \in K\} = \{\langle k_i, \gamma_i^k, \delta_i^k \rangle | 1 \leq i \leq m\}$ and

$$\langle \gamma_i^k, \delta_i^k \rangle = \begin{cases} \langle \alpha_i^k, \beta_i^k \rangle, & \text{if } \langle \alpha_i^k, \beta_i^k \rangle > \langle \alpha, \beta \rangle \\ \langle 0, 1 \rangle, & \text{otherwise} \end{cases};$$

$$\mathcal{L}_{\overline{R}, \alpha, \beta}^>(A) = [\overline{K}^*, L^*, \{\langle \mu_{k_i, l_j}, \nu_{k_i, l_j} \rangle\}],$$

$$\text{where } \langle \gamma_i^k, \delta_i^k \rangle = \begin{cases} \langle \alpha_i^k, \beta_i^k \rangle, & \text{if } \langle \alpha_i^k, \beta_i^k \rangle \geq \langle \alpha, \beta \rangle \\ \langle 0, 1 \rangle, & \text{otherwise} \end{cases};$$

$$\mathcal{L}_{R, \alpha, \beta}^<(A) = [\overline{K}^*, L^*, \{\langle \mu_{k_i, l_j}, \nu_{k_i, l_j} \rangle\}],$$

$$\text{where } \langle \gamma_i^k, \delta_i^k \rangle = \begin{cases} \langle \alpha_i^k, \beta_i^k \rangle, & \text{if } \langle \alpha_i^k, \beta_i^k \rangle < \langle \alpha, \beta \rangle \\ \langle 1, 0 \rangle, & \text{otherwise} \end{cases};$$

$$\mathcal{L}_{\overline{R}, \alpha, \beta}^<(A) = [\overline{K}^*, L^*, \{\langle \mu_{k_i, l_j}, \nu_{k_i, l_j} \rangle\}],$$

$$\text{where } \langle \gamma_i^k, \delta_i^k \rangle = \begin{cases} \langle \alpha_i^k, \beta_i^k \rangle, & \text{if } \langle \alpha_i^k, \beta_i^k \rangle \leq \langle \alpha, \beta \rangle \\ \langle 1, 0 \rangle, & \text{otherwise} \end{cases};$$

$$\mathcal{L}_{C, \alpha, \beta}^>(A) = [K^*, \overline{L}^*, \{\langle \mu_{k_i, l_j}, \nu_{k_i, l_j} \rangle\}],$$

where here and below $\overline{L}^* = \{\langle k_i, \gamma_i^l, \delta_i^l \rangle | k_i \in K\} = \{\langle k_i, \gamma_i^l, \delta_i^l \rangle | 1 \leq i \leq m\}$ and

$$\langle \gamma_i^l, \delta_i^l \rangle = \begin{cases} \langle \alpha_i^l, \beta_i^l \rangle, & \text{if } \langle \alpha_i^l, \beta_i^l \rangle > \langle \alpha, \beta \rangle \\ \langle 0, 1 \rangle, & \text{otherwise} \end{cases};$$

$$\mathcal{L}_{C,\alpha,\beta}^{\geq}(A) = [K^*, \bar{L}^*, \{\langle \mu_{k_i, l_j}, \nu_{k_i, l_j} \rangle\}],$$

$$\text{where } \langle \gamma_i^l, \delta_i^l \rangle = \begin{cases} \langle \alpha_i^l, \beta_i^l \rangle, & \text{if } \langle \alpha_i^l, \beta_i^l \rangle \geq \langle \alpha, \beta \rangle \\ \langle 0, 1 \rangle, & \text{otherwise} \end{cases};$$

$$\mathcal{L}_{C,\alpha,\beta}^{<}(A) = [K^*, \bar{L}^*, \{\langle \mu_{k_i, l_j}, \nu_{k_i, l_j} \rangle\}],$$

$$\text{where } \langle \gamma_i^l, \delta_i^l \rangle = \begin{cases} \langle \alpha_i^l, \beta_i^l \rangle, & \text{if } \langle \alpha_i^l, \beta_i^l \rangle < \langle \alpha, \beta \rangle \\ \langle 1, 0 \rangle, & \text{otherwise} \end{cases};$$

$$\mathcal{L}_{C,\alpha,\beta}^{\leq}(A) = [K^*, \bar{L}^*, \{\langle \mu_{k_i, l_j}, \nu_{k_i, l_j} \rangle\}],$$

$$\text{where } \langle \gamma_i^l, \delta_i^l \rangle = \begin{cases} \langle \alpha_i^l, \beta_i^l \rangle, & \text{if } \langle \alpha_i^l, \beta_i^l \rangle \leq \langle \alpha, \beta \rangle \\ \langle 1, 0 \rangle, & \text{otherwise} \end{cases};$$

Now, Theorem 1 of [53] preserves its sense when we change operator $\mathcal{L}_{\alpha,\beta}^z$ with any of the operators $\mathcal{L}_{R,\alpha,\beta}^z$ and $\mathcal{L}_{C,\alpha,\beta}^z$, where $z \in \{>, \geq, <, \leq\}$. Moreover, the next assertion is valid: for each EIFIM A and for every two IFPs $\langle \alpha, \beta \rangle, \langle \gamma, \delta \rangle$:

$$\mathcal{L}_{R,\alpha,\beta}^z(\mathcal{L}_{C,\alpha,\beta}^y(A)) = \mathcal{L}_{C,\alpha,\beta}^y(\mathcal{L}_{R,\alpha,\beta}^z(A)),$$

where $y, z \in \{>, \geq, <, \leq\}$.

Theorem 2 from [53] also preserves its sense for each of operations $\oplus_\circ, \otimes_\circ, \odot_\circ, *$ when we change operator $\mathcal{L}_{\alpha,\beta}^z$ with any of the operators $\mathcal{L}_{R,\alpha,\beta}^z$ and $\mathcal{L}_{C,\alpha,\beta}^z$, where $z \in \{>, \geq, <, \leq\}$, and relations $\subseteq_v$ and $\supseteq_v$ with relations $\subseteq_d$ and $\supseteq_d$, respectively.

Finally, we modify again the operator **automatic reduction** to the forms:

$$@_R(A) = [P^*, L^*, \{b_{p_r, q_s}\}],$$

where $P^* \subseteq K^*$ are index sets with the following property:

$$(\forall k \in K - P)(\langle \alpha_i^k, \beta_i^k \rangle = \langle 0, 1 \rangle) \ \& \ (\forall p_r = k_i \in P)(\forall q_s = l_j \in L)(b_{p_r, q_s} = a_{k_i, l_j}),$$

$$@_C(A) = [K^*, Q^*, \{b_{p_r, q_s}\}],$$

where $Q^* \subseteq L^*$ are index sets with the following property:

$$(\forall l \in L - Q)(\langle \alpha_i^l, \beta_i^l \rangle = \langle 0, 1 \rangle) \ \& \ (\forall q_s = l_j \in Q)(\forall p_r = k_i \in K)(b_{p_r, q_s} = a_{k_i, l_j}).$$

Finally, we would like to mention that IFSs have other extensions that can also be the basis for new types of IFIMs. Such extensions are, for example, Circular IFSs [33], Elliptic IFSs [34], IFS of the second and n -th type [7], multidimensional IFSs [13], and others.

Interval-Valued IFIMs (IVIFIMs)

5.1 Definition of an IVIFIM

The results in this section were originally introduced in [35]. Here, we will follow—and correct—some results from this paper.

Let I be a fixed set. By an Interval-Valued Intuitionistic Fuzzy Index Matrix (IVIFIM) with index sets K and L ($K, L \subset I$), we denote the object:

$$[K, L, \{\langle M_{k_i, l_j}, N_{k_i, l_j} \rangle\}]$$

$$\equiv \begin{array}{c|cccc} & l_1 & \dots & l_j & \dots & l_n \\ \hline k_1 & \langle M_{k_1, l_1}, N_{k_1, l_1} \rangle & \dots & \langle M_{k_1, l_j}, N_{k_1, l_j} \rangle & \dots & \langle M_{k_1, l_n}, N_{k_1, l_n} \rangle \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_i & \langle M_{k_i, l_1}, N_{k_i, l_1} \rangle & \dots & \langle M_{k_i, l_j}, N_{k_i, l_j} \rangle & \dots & \langle M_{k_i, l_n}, N_{k_i, l_n} \rangle \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_m & \langle M_{k_m, l_1}, N_{k_m, l_1} \rangle & \dots & \langle M_{k_m, l_j}, N_{k_m, l_j} \rangle & \dots & \langle M_{k_m, l_n}, N_{k_m, l_n} \rangle \end{array},$$

where for every $1 \leq i \leq m, 1 \leq j \leq n$: $M_{k_i, l_j}, N_{k_i, l_j} \subseteq [0, 1]$, $\sup M_{k_i, l_j} + \sup N_{k_i, l_j} \leq 1$ and

$$K = \{k_1, k_2, \dots, k_m\},$$

$$L = \{l_1, l_2, \dots, l_n\}.$$

Now, for the above sets K and L , the Extended IVIFIM (EIVIFIM) is defined by:

$$\begin{array}{c}
[K^*, L^*, \{\langle M_{k_i, l_j}, N_{k_i, l_j} \rangle\}] \\
\hline
\begin{array}{c}
k_1, \langle A_{k_1}, B_{k_1} \rangle \\
\vdots \\
k_i, \langle A_{k_i}, B_{k_i} \rangle \\
\vdots \\
k_m, \langle A_{k_m}, B_{k_m} \rangle
\end{array}
\left| \begin{array}{ccccc}
l_1, \langle A_{l_1}, B_{l_1} \rangle & \dots & l_j, \langle A_{l_j}, B_{l_j} \rangle & \dots & l_n, \langle A_{l_n}, B_{l_n} \rangle \\
\langle M_{k_1, l_1}, N_{k_1, l_1} \rangle & \dots & \langle M_{k_1, l_j}, N_{k_1, l_j} \rangle & \dots & \langle M_{k_1, l_n}, N_{k_1, l_n} \rangle \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
\langle M_{k_i, l_1}, N_{k_i, l_1} \rangle & \dots & \langle M_{k_i, l_j}, N_{k_i, l_j} \rangle & \dots & \langle M_{k_i, l_n}, N_{k_i, l_n} \rangle \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
\langle M_{k_m, l_1}, N_{k_m, l_1} \rangle & \dots & \langle M_{k_m, l_j}, N_{k_m, l_j} \rangle & \dots & \langle M_{k_m, l_n}, N_{k_m, l_n} \rangle
\end{array} \right.
\end{array}$$

where for every $1 \leq i \leq m, 1 \leq j \leq n$:

$$\begin{aligned}
M_{k_i, l_j}, N_{k_i, l_j} &\subseteq [0, 1], \sup M_{k_i, l_j} + \sup N_{k_i, l_j} \leq 1, \\
A_{k_i}, B_{k_i} &\subseteq [0, 1], \sup A_{k_i} + \sup B_{k_i} \leq 1, \\
A_{l_j}, B_{l_j} &\subseteq [0, 1], \sup A_{l_j} + \sup B_{l_j} \leq 1
\end{aligned}$$

and here and below,

$$\begin{aligned}
K^* &= \{\langle k_i, A_{k_i}, B_{k_i} \rangle | k_i \in K\} = \{\langle k_i, A_{k_i}, B_{k_i} \rangle | 1 \leq i \leq m\}, \\
L^* &= \{\langle l_j, A_{l_j}, B_{l_j} \rangle | l_j \in L\} = \{\langle l_j, A_{l_j}, B_{l_j} \rangle | 1 \leq j \leq n\}.
\end{aligned}$$

Let

$$\begin{aligned}
K^* \subset P^* &\text{ iff } (K \subset P) \ \& \ (\forall k_i = p_i \in K)((A_{k_i} \subset^* A_{p_i}) \ \& \ (B_{k_i} \supset^* B_{p_i})), \\
K^* \subseteq P^* &\text{ iff } (K \subseteq P) \ \& \ (\forall k_i = p_i \in K)((A_{k_i} \subseteq A_{p_i}) \ \& \ (B_{k_i} \supseteq B_{p_i})),
\end{aligned}$$

where for two arbitrary sets X and Y :

$$\begin{aligned}
X \subset^* Y &\text{ iff } \inf X < \inf Y \text{ and } \sup X < \sup Y, \\
X \supset^* Y &\text{ iff } \inf X > \inf Y \text{ and } \sup X > \sup Y, \\
X \subseteq^* Y &\text{ iff } \inf X \leq \inf Y \text{ and } \sup X \leq \sup Y, \\
X \supseteq^* Y &\text{ iff } \inf X \geq \inf Y \text{ and } \sup X \geq \sup Y.
\end{aligned}$$

We must note that the above relations are marked with “*” to differentiate them from the standard set-theoretical relations $\subset, \supset, \subseteq, \supseteq$.

Below, we will work only with EIVIFIMs, because the IVIFIMs are partial their cases.

The pair $\langle M, N \rangle$ is an Interval-Valued Intuitionistic Fuzzy Pair (IVIFP) if and only if $M, N \subseteq [0, 1]$ and $\sup M + \sup N \leq 1$. For example, there the first interval-valued intuitionistic fuzzy implication (called first Zadeh’s interval-valued intuitionistic fuzzy implication) of IVIFPs $\langle M, N \rangle$ and $\langle P, Q \rangle$ has the form:

$$\begin{aligned}
x \rightarrow_1 y &= [\max(\inf N, \min(\inf M, \inf P)), \max(\sup N, \min(\sup M, \sup P))], \\
&\quad [\min(\inf M, \inf Q), \min(\sup M, \sup Q)].
\end{aligned}$$

In the same manner, we can construct the interval-valued intuitionistic fuzzy conjunctions and disjunctions. For example, the following intuitionistic fuzzy conjunctions and disjunctions can be introduced (cf. [26]):

$$\begin{aligned}
x \wedge_4 y = x \wedge y &= \langle [\min(\inf M, \inf P), \min(\sup M, \sup N)], \\
&\quad [\max(\inf N, \inf Q), \max(\sup N, \sup Q)] \rangle \\
x \vee_4 y = x \vee y &= \langle [\max(\inf M, \inf P), \max(\sup M, \sup N)], \\
&\quad [\min(\inf N, \inf Q), \min(\sup N, \sup Q)] \rangle \\
x \wedge_2 y = x + y &= \langle [\inf M + \inf P - \inf M \inf P, \sup M + \sup P - \sup M \sup P], \\
&\quad [\inf N \inf Q, \sup N \sup Q] \rangle \\
x \vee_2 y = x.y &= \langle [\inf M \inf P, \sup M \sup P], \\
&\quad [\inf N + \inf Q - \inf N \inf Q, \sup N + \sup Q - \sup N \sup Q] \rangle
\end{aligned}$$

Let “ $*$ ” and “ $\circ$ ” be two fixed operations over IVIFPs and let

$$\begin{aligned}
\langle M, N \rangle * \langle P, Q \rangle &= \langle M *_l P, N *_r Q \rangle, \\
\langle M, N \rangle \circ \langle P, Q \rangle &= \langle M \circ_l P, N \circ_r Q \rangle,
\end{aligned}$$

where the forms of operations “ $*_l$ ” and “ $*_r$ ”, and operations “ $\circ_l$ ” and “ $\circ_r$ ” are determined by the form of the operation “ $*$ ” and “ $\circ$ ”, respectively.

For the EIVIFIMs

$$\begin{aligned}
A &= [K^*, L^*, \{\langle M_{k_i, l_j}, N_{k_i, l_j} \rangle\}], \\
B &= [P^*, Q^*, \{\langle R_{p_r, q_s}, S_{p_r, q_s} \rangle\}],
\end{aligned}$$

with

$$\begin{aligned}
K^* &= \{\langle k_i, A_{k_i}, B_{k_i} \rangle | k_i \in K\}, \\
L^* &= \{\langle l_j, A_{l_j}, B_{l_j} \rangle | l_j \in L\}, \\
P^* &= \{\langle p_r, C_{p_r}, D_{q_s} \rangle | p_r \in P\}, \\
Q^* &= \{\langle q_s, C_{p_r}, D_{q_s} \rangle | q_s \in Q\}
\end{aligned}$$

operations that are analogous to the usual matrix operations of addition and multiplication are defined, related to IVIF-nature of the presented IMs.

5.2 Standard operations over IVIFIMs

Below, we will discuss the operations over IVIFIMs similarly to the previous chapters.

Addition- $(*)$

$$A \oplus_{(*)} B = [T^*, V^*, \{\langle \Phi_{t_u, v_w}, \Psi_{t_u, v_w} \rangle\}],$$

where

$$\begin{aligned}
T^* &= K^* \cup P^* = \{\langle t_u, A_{t_u}, B_{t_u} \rangle | t_u \in K \cup P\}, \\
V^* &= L^* \cup Q^* = \{\langle v_w, A_{v_w}, B_{v_w} \rangle | v_w \in L \cup Q\}, \\
A_{t_u} &= \begin{cases} A_{k_i}, & \text{if } t_u = k_i \in K - P; \\ A_{p_r}, & \text{if } t_u = p_r \in P - K; \\ A_{k_i} \cup A_{p_r}, & \text{if } t_u = k_i = p_r \in K \cap P, \end{cases} \\
B_{v_w} &= \begin{cases} B_{l_j}, & \text{if } v_w = l_j \in L - Q; \\ B_{q_s}, & \text{if } v_w = q_s \in Q - L; \\ B_{v_w} \cap B_{q_s}, & \text{if } v_w = l_j = q_s \in L \cap Q, \end{cases}
\end{aligned}$$

and

$$\langle \Phi_{t_u, v_w}, \Psi_{t_u, v_w} \rangle = \begin{cases} \langle M_{k_i, l_j}, N_{k_i, l_j} \rangle, & \text{if } t_u = k_i \in K \text{ and } v_w = l_j \in L - Q; \\ & \text{or } t_u = k_i \in K - P \text{ and } v_w = l_j \in L; \\ \langle R_{p_r, q_s}, S_{p_r, q_s} \rangle, & \text{if } t_u = p_r \in P \text{ and } v_w = q_s \in Q - L \\ & \text{or } t_u = p_r \in P - K \text{ and } v_w = q_s \in Q; \\ \langle M_{k_i, l_j} *_l R_{p_r, q_s}, N_{k_i, l_j} *_r S_{p_r, q_s} \rangle, & \text{if } t_u = k_i = p_r \in K \cap P \\ & \text{and } v_w = l_j = q_s \in L \cap Q \\ \langle [0, 0], [1, 1] \rangle, & \text{otherwise.} \end{cases}$$

Termwise multiplication-(*)

$$A \otimes_{(*)} B = [T^*, V^*, \{\langle \Phi_{t_u, v_w}, \Psi_{t_u, v_w} \rangle\}],$$

where

$$\begin{aligned}
T^* &= K^* \cap P^* = \{\langle t_u, A_{t_u}, B_{t_u} \rangle | t_u \in K \cup P\}, \\
V^* &= L^* \cap Q^* = \{\langle v_w, A_{v_w}, B_{v_w} \rangle | v_w \in L \cup Q\}, \\
A_{t_u} &= A_{k_i} \cap A_{p_r}, \text{ for } t_u = k_i = p_r \in K \cap P, \\
B_{v_w} &= B_{v_w} \cup B_{q_s}, \text{ for } v_w = l_j = q_s \in L \cap Q,
\end{aligned}$$

and

$$\langle \Phi_{t_u, v_w}, \Psi_{t_u, v_w} \rangle = \langle M_{k_i, l_j} *_l R_{p_r, q_s}, N_{k_i, l_j} *_r S_{p_r, q_s} \rangle.$$

Multiplication-(o, *)

$$A \odot_{(o, *)} B = [T^*, V^*, \{\langle \Phi_{t_u, v_w}, \Psi_{t_u, v_w} \rangle\}],$$

where

$$\begin{aligned}
T^* &= (K \cup (P - L))^* = \{\langle t_u, A_{t_u}, B_{t_u} \rangle | t_u \in K \cup (P - L)\}, \\
V^* &= (Q \cup (L - P))^* = \{\langle v_w, A_{v_w}, B_{v_w} \rangle | v_w \in Q \cup (L - P)\},
\end{aligned}$$

$$A_{t_u} = \begin{cases} A_{k_i}, & \text{if } t_u = k_i \in K, \\ A_{p_r}, & \text{if } t_u = p_r \in P - L, \end{cases} \quad B_{v_w} = \begin{cases} B_{l_j}, & \text{if } v_w = l_j \in L - P, \\ B_{q_s}, & \text{if } v_w = q_s \in Q. \end{cases}$$

$$= \begin{cases} \langle \Phi_{t_u, v_w}, \Psi_{t_u, v_w} \rangle & \\ \langle M_{k_i, l_j}, N_{k_i, l_j} \rangle, & \text{if } t_u = k_i \in K \\ & \text{and } v_w = l_j \in L - P - Q; \\ \langle R_{p_r, q_s}, S_{p_r, q_s} \rangle, & \text{if } t_u = p_r \in P - L - K \\ & \text{and } v_w = q_s \in Q; \\ \langle \underset{\in L \cap P}{l_j \underset{=p_r}{\circ}_l} (M_{k_i, l_j} *_l R_{p_r, q_s}), \underset{\in L \cap P}{l_j \underset{=p_r}{\circ}_r} (N_{k_i, l_j} *_r S_{p_r, q_s}) \rangle, & \text{if } t_u = k_i \in K \\ & \text{and } v_w = q_s \in Q; \\ \langle [0, 0], [1, 1] \rangle, & \text{otherwise} \end{cases}$$

Structural subtraction

$$A \ominus B = [T^*, V^*, \{\langle \Phi_{t_u, v_w}, \Psi_{t_u, v_w} \rangle\}],$$

where

$$T^* = (K - P)^* = \{\langle t_u, A_{t_u}, B_{t_u} \rangle | t_u \in K - P\},$$

$$V^* = (L - Q)^* = \{\langle v_w, A_{v_w}, B_{v_w} \rangle | v_w \in L - Q\},$$

for the set-theoretic subtraction operation and

$$A_{t_u} = A_{k_i}, \text{ for } t_u = k_i \in K - P,$$

$$B_{v_w} = B_{l_j}, \text{ for } v_w = l_j \in L - Q$$

and $\langle \Phi_{t_u, v_w}, \Psi_{t_u, v_w} \rangle = \langle M_{k_i, l_j}, N_{k_i, l_j} \rangle$, for $t_u = k_i \in K - P$ and $v_w = l_j \in L - Q$.

Negation of an EIVIFIM

$$\neg A = [T^*, V^*, \{\neg \langle M_{k_i, l_j}, N_{k_i, l_j} \rangle\}],$$

where $\neg$ is one of the possible interval-valued intuitionistic fuzzy negations that can be defined on the basis of the definitions of intuitionistic fuzzy negations.

5.3 Relations over IVIFIMs

Let the two EIVIFIMs $A = [K^*, L^*, \{\langle A_{k,l}, B_{k,l} \rangle\}]$ and $B = [P^*, Q^*, \{\langle C_{p,q}, D_{p,q} \rangle\}]$ be given. We shall introduce the following definitions where $\subset$ and $\subseteq$ denote the relations “strong inclusion” and “weak inclusion”.

The strict relation “inclusion about dimension” is

$$\begin{aligned} A \subset_d B \text{ iff } & (((K^* \subset P^*) \& (L^* \subset Q^*)) \vee ((K^* \subseteq P^*) \& (L^* \subset Q^*)) \\ & \vee ((K^* \subset P^*) \& (L^* \subseteq Q^*))) \& (\forall k \in K)(\forall l \in L) \\ & (\langle A_{k,l}, B_{k,l} \rangle = \langle C_{k,l}, D_{k,l} \rangle). \end{aligned}$$

The non-strict relation “inclusion about dimension” is

$$A \subseteq_d B \text{ iff } (K^* \subseteq P^*) \ \& \ (L^* \subseteq Q^*) \\ \& \ (\forall k \in K)(\forall l \in L)(\langle A_{k,l}, B_{k,l} \rangle = \langle C_{k,l}, D_{k,l} \rangle).$$

The strict relation “inclusion about value” is

$$A \subset_v B \text{ iff } (K^* = P^*) \ \& \ (L^* = Q^*) \\ \& \ (\forall k \in K)(\forall l \in L)(\langle A_{k,l}, B_{k,l} \rangle \subset^* \langle C_{k,l}, D_{k,l} \rangle).$$

The non-strict relation “inclusion about value” is

$$A \subseteq_v B \text{ iff } (K^* = P^*) \ \& \ (L^* = Q^*) \\ \& \ (\forall k \in K)(\forall l \in L)(\langle A_{k,l}, B_{k,l} \rangle \subseteq \langle C_{k,l}, D_{k,l} \rangle).$$

The strict relation “inclusion” is

$$A \subset_* B \text{ iff } (((K^* \subset P^*) \ \& \ (L^* \subset Q^*)) \vee ((K^* \subseteq P^*) \ \& \ (L^* \subset Q^*))) \\ \vee ((K^* \subset P^*) \ \& \ (L^* \subseteq Q^*)) \ \& \ (\forall k \in K)(\forall l \in L) \\ (\langle A_{k,l}, B_{k,l} \rangle \subset^* \langle C_{k,l}, D_{k,l} \rangle).$$

The non-strict relation “inclusion” is

$$A \subseteq_* B \text{ iff } (K^* \subseteq P^*) \ \& \ (L^* \subseteq Q^*) \ \& \ (\forall k \in K)(\forall l \in L) \\ (\langle A_{k,l}, B_{k,l} \rangle \subseteq \langle C_{k,l}, D_{k,l} \rangle).$$

The strict relation “inclusion about first index values” is

$$A \subset_{i,f} B \text{ iff } (K^* = P^*) \ \& \ (\forall k_i = p_q \in K)((\alpha_i^k < \alpha_q^p) \ \& \ (\beta_i^k > \beta_q^p)).$$

The non-strict relation “inclusion about first index values” is

$$A \subseteq_{i,f} B \text{ iff } (K^* = P^*) \ \& \ (\forall k_i = p_q \in K)((\alpha_i^k \leq \alpha_q^p) \ \& \ (\beta_i^k \geq \beta_q^p)).$$

The strict relation “inclusion about second index values” is

$$A \subset_{i,s} B \text{ iff } (L^* = Q^*) \ \& \ (\forall l_j = r_s \in L)((\alpha_j^l < \alpha_s^r) \ \& \ (\beta_j^l > \beta_s^r)).$$

The non-strict relation “inclusion about first index values” is

$$A \subseteq_{i,s} B \text{ iff } (L^* = Q^*) \ \& \ (\forall l_j = r_s \in L)((\alpha_j^l \leq \alpha_s^r) \ \& \ (\beta_j^l \geq \beta_s^r)).$$

5.4 Other operations over IVIFIMs

Let the EIVIFIM A be given and let $k_0 \notin K$ and $l_0 \notin L$ be two fixed indices.

Now, we introduce the following two operations over it having a general form.

Let “ $*$ ” be a fixed operation over IVIFPs.

(*)-row-aggregation

$$\rho_{(*)}(A, k_0) = \frac{k_0, \left\langle \begin{smallmatrix} *l \\ 1 \leq i \leq m \end{smallmatrix} A_{k_i}, \begin{smallmatrix} *r \\ 1 \leq i \leq m \end{smallmatrix} B_{k_i} \right\rangle}{\left| \begin{array}{c} l_1, \langle A_{l_1}, B_{l_1} \rangle \quad \dots \\ \dots \quad l_n, \langle A_{l_n}, B_{l_n} \rangle \\ \dots \left\langle \begin{smallmatrix} *l \\ 1 \leq i \leq m \end{smallmatrix} M_{k_i, l_n}, \begin{smallmatrix} *r \\ 1 \leq i \leq m \end{smallmatrix} N_{k_i, l_n} \end{smallmatrix} \right\rangle \end{array} \right.}$$

(*)-column-aggregation

$$\sigma_{(*)}(A, l_0) = \frac{l_0, \left\langle \begin{smallmatrix} *l \\ 1 \leq i \leq m \end{smallmatrix} A_{l_j}, \begin{smallmatrix} *r \\ 1 \leq i \leq m \end{smallmatrix} B_{l_j} \right\rangle}{\left| \begin{array}{c} k_1, \langle A_{k_1}, B_{k_1} \rangle \\ \vdots \\ k_m, \langle A_{k_m}, B_{k_m} \rangle \end{array} \right| \left\langle \begin{smallmatrix} *l \\ 1 \leq i \leq m \end{smallmatrix} M_{k_i, l_j}, \begin{smallmatrix} *r \\ 1 \leq i \leq m \end{smallmatrix} N_{k_i, l_j} \right\rangle}.$$

Here and below we use the symbol “ $\perp$ ” indicating lack of some component in the separate definitions. In some cases, it is suitable to change this symbol with “ $\langle [0, 0], [1, 1] \rangle$ ”. This is realized by some of the operations “reduction” over an EIVIFIM.

First, we introduce the operations $(k, \perp)$ - and $(\perp, l)$ -reduction of a given IVIFIM A :

$$A_{(k, \perp)} = [(K - \{k\})^*, L^*, \{\langle P_{t_u, v_w}, Q_{t_u, v_w} \rangle\}]$$

where $\langle P_{t_u, v_w}, Q_{t_u, v_w} \rangle = \langle M_{k_i, l_j}, N_{k_i, l_j} \rangle$ for $t_u = k_i \in K - \{k\}$ and $v_w = l_j \in L$ and

$$A_{(\perp, l)} = [K^*, (L - \{l\})^*, \{\langle P_{t_u, v_w}, Q_{t_u, v_w} \rangle\}],$$

where $\langle P_{t_u, v_w}, Q_{t_u, v_w} \rangle = \langle M_{k_i, l_j}, N_{k_i, l_j} \rangle$ for $t_u = k_i \in K$ and $v_w = l_j \in L - \{l\}$.

Second, we define

$$A_{(k,l)} = (A_{(k,\perp)})_{(\perp,l)} = (A_{(\perp,l)})_{(k,\perp)}.$$

Third, let $R = \{k_1, k_2, \dots, k_s\} \subseteq K$ and $S = \{q_1, q_2, \dots, q_t\} \subseteq L$. Then, we define the following three operations:

$$\begin{aligned} A_{(R,l)} &= (\cdots ((A_{(k_1,l)})_{(k_2,l)}) \cdots)_{(k_s,l)}, \\ A_{(k,S)} &= (\cdots ((A_{(k,l_1)})_{(k,l_2)}) \cdots)_{(k,l_t)}, \\ A_{(R,S)} &= (\cdots ((A_{(p_1,S)})_{(p_2,S)}) \cdots)_{(p_s,S)} = (\cdots ((A_{(R,q_1)})_{(R,q_2)}) \cdots)_{(R,q_t)}. \end{aligned}$$

Let $M \subseteq K$ and $N \subseteq L$ and A be an EIVIFIM. Then, operation “projection” over an EIVIFIM has the form

$$\text{pr}_{M,N}A = [M, N, \{\langle P_{t_u, v_w}, Q_{t_u, v_w} \rangle\}],$$

where for each $k_i \in M$ and each $l_j \in N$,

$$\langle P_{k_i, l_j}, Q_{k_i, l_j} \rangle = \langle M_{k_i, l_j}, N_{k_i, l_j} \rangle.$$

Obviously, for every EIVIFIM A and for the sets $M_1 \subseteq M_2 \subseteq K$ and $N_1 \subseteq N_2 \subseteq L$ the equality

$$\text{pr}_{M_1, N_1} \text{pr}_{M_2, N_2} A = \text{pr}_{M_1, N_1} A$$

is valid and for $M \subseteq K, N \subseteq L$, and the following equalities hold true:

$$\begin{aligned} \text{pr}_{M,N}A &= A_{(K-M, L-N)}, \\ A_{M,N} &= \text{pr}_{K-M, L-N}A. \end{aligned}$$

5.5 Modal and level operators over IVIFIMs

Below, we will discuss the possibility to apply some of the operators defined over IVIFSs, but now, over EIVIFIMs. Some of the basic operators over IVIFSs are the following (see [31]):

Modal operators

$$\begin{aligned} \Box A &= \{\langle x, M_A(x), [\inf N_A(x), 1 - \sup M_A(x)] \rangle \mid x \in E\}, \\ \Diamond A &= \{\langle x, [\inf M_A(x), 1 - \sup N_A(x)], N_A(x) \rangle \mid x \in E\}, \\ D_\alpha(A) &= \{\langle x, [\inf M_A(x), \sup M_A(x) + \alpha.(1 - \sup M_A(x) - \sup N_A(x))], \\ &\quad [\inf N_A(x), \sup N_A(x) + (1 - \alpha).(1 - \sup M_A(x) - \sup N_A(x))] \rangle \mid x \in E\}, \end{aligned}$$

$$\begin{aligned}
F_{\alpha,\beta}(A) &= \{ \langle x, [\inf M_A(x), \sup M_A(x) + \alpha.(1 - \sup M_A(x) - \sup N_A(x))], \\
&\quad [\inf N_A(x), \sup N_A(x) + \beta.(1 - \sup M_A(x) - \sup N_A(x))] \rangle \\
&\quad \mid x \in E \}, \text{ for } \alpha + \beta \leq 1, \\
G_{\alpha,\beta}(A) &= \{ \langle x, [\alpha. \inf M_A(x), \alpha. \sup M_A(x)], [\beta. \inf N_A(x), \beta. \sup N_A(x)] \rangle \\
&\quad \mid x \in E \}, \\
H_{\alpha,\beta}(A) &= \{ \langle x, [\alpha. \inf M(x), \alpha. \sup M_A(x)], [\inf N_A(x), \\
&\quad \sup N_A(x) + \beta.(1 - \sup M_A(x) - \sup N_A(x))] \rangle \mid x \in E \}, \\
H_{\alpha,\beta}^*(A) &= \{ \langle x, [\alpha. \inf M_A(x), \alpha. \sup M_A(x)], [\inf N_A(x), \sup N_A(x) \\
&\quad + \beta.(1 - \alpha. \sup M_A(x) - \sup N_A(x))] \rangle \mid x \in E \}, \\
J_{\alpha,\beta}(A) &= \{ \langle x, [\inf M_A(x), \sup M_A(x) + \alpha.(1 - \sup M_A(x) - \sup N_A(x))], \\
&\quad [\beta. \inf N_A(x), \beta. \sup N_A(x)] \rangle \mid x \in E \}, \\
J_{\alpha,\beta}^*(A) &= \{ \langle x, [\inf M_A(x), \sup M_A(x) + \alpha.(1 - \sup M_A(x) \\
&\quad - \beta. \sup N_A(x))], [\beta. \inf N_A(x), \beta. \sup N_A(x)] \rangle \mid x \in E \},
\end{aligned}$$

where $\alpha, \beta \in [0, 1]$.

$$\begin{aligned}
\overline{F}_{\alpha,\beta,\gamma,\delta}(A) &= \{ \langle x, [\inf M_A(x) + \alpha.(1 - \sup M_A(x) - \sup N_A(x)), \\
&\quad \sup M_A(x) + \beta.(1 - \sup M_A(x) - \sup N_A(x))], \\
&\quad [\inf N_A(x) + \gamma.(1 - \sup M_A(x) - \sup N_A(x)), \\
&\quad \sup N_A(x) + \delta.(1 - \sup M_A(x) - \sup N_A(x))] \rangle \mid x \in E \}
\end{aligned}$$

where $\beta + \delta \leq 1$,

$$\begin{aligned}
\overline{G}_{\alpha,\beta,\gamma,\delta}(A) &= \{ \langle x, [\alpha. \inf M_A(x), \beta. \sup M_A(x)], \\
&\quad [\gamma. \inf N_A(x), \delta. \sup N_A(x)] \rangle \mid x \in E \}, \\
\overline{H}_{\alpha,\beta,\gamma,\delta}(A) &= \{ \langle x, [\alpha. \inf M_A(x), \beta. \sup M_A(x)], \\
&\quad [\inf N_A(x) + \gamma.(1 - \sup M_A(x) - \sup N_A(x)), \\
&\quad \sup N_A(x) + \delta.(1 - \sup M_A(x) - \sup N_A(x))] \rangle \mid x \in E \} \\
\overline{H}_{\alpha,\beta,\gamma,\delta}^*(A) &= \{ \langle x, [\alpha. \inf M_A(x), \beta. \sup M_A(x)], \\
&\quad [\inf N_A(x) + \gamma.(1 - \beta. \sup M_A(x) - \sup N_A(x)), \\
&\quad \sup N_A(x) + \delta.(1 - \beta. \sup M_A(x) - \sup N_A(x))] \rangle \mid x \in E \} \\
\overline{J}_{\alpha,\beta,\gamma,\delta}(A) &= \{ \langle x, [\inf M_A(x) + \alpha.(1 - \sup M_A(x) - \sup N_A(x)), \\
&\quad \sup M_A(x) + \beta.(1 - \sup M_A(x) - \sup N_A(x))], \\
&\quad [\gamma. \inf N_A(x), \delta. \sup N_A(x)] \rangle \mid x \in E \}
\end{aligned}$$

$$\begin{aligned} \bar{J}_{\alpha,\beta,\gamma,\delta}^*(A) = \{ \langle x, [\inf M_A(x) + \alpha.(1 - \delta.\sup M_A(x) - \sup N_A(x)), \\ \sup M_A(x) + \beta.(1 - \sup M_A(x) - \delta.\sup N_A(x))], \\ [\gamma.\inf N_A(x), \delta.\sup N_A(x)] \rangle \mid x \in E \} \end{aligned}$$

where $\alpha, \beta, \gamma, \delta \in [0, 1]$ such that $\alpha \leq \beta$ and $\gamma \leq \delta$;

Level operators

$$\begin{aligned} P_{\alpha,\beta}(A) &= \{ \langle x, [\max(\alpha, \inf M_A(x)), \max(\alpha, \sup M_A(x))], \\ &\quad [\min(\beta, \inf N_A(x)), \min(\beta, \sup N_A(x))] \rangle \mid x \in E \}, \\ Q_{\alpha,\beta}(A) &= \{ \langle x, [\min(\alpha, \inf M_A(x)), \min(\alpha, \sup M_A(x))], \\ &\quad [\max(\beta, \inf N_A(x)), \max(\beta, \sup N_A(x))] \rangle \mid x \in E \}, \end{aligned}$$

where $\alpha, \beta \in [0, 1]$ and $\alpha + \beta \leq 1$

$$\begin{aligned} P_{\alpha,\beta,\gamma,\delta}(A) &= \{ \langle x, [\max(\alpha, \inf M_A(x)), \max(\beta, \sup M_A(x))], \\ &\quad [\min(\gamma, \inf N_A(x)), \min(\delta, \sup N_A(x))] \rangle \mid x \in E \}, \\ Q_{\alpha,\beta,\gamma,\delta}(A) &= \{ \langle x, [\min(\alpha, \inf M_A(x)), \min(\beta, \sup M_A(x))], \\ &\quad [\max(\gamma, \inf N_A(x)), \max(\delta, \sup N_A(x))] \rangle \mid x \in E \}, \end{aligned}$$

where $\alpha, \beta, \gamma, \delta \in [0, 1]$ such that $\alpha \leq \beta$, $\gamma \leq \delta$ and $\beta + \delta \leq 1$:

Let, as above, $x = \langle M, N \rangle$ be an IVIFP and let O_X^1, O_Y^2, O_Z^3 be three interval-valued intuitionistic fuzzy operators with parameters X, Y, Z , respectively. In practice, the parameters X, Y, Z can be IFPs or IVIFPs.

The three operators affect the K -, L -indices and the $\langle M_{k_i, l_j}, N_{k_i, l_j} \rangle$ -elements, respectively. They can be applied to an EIVIFIM A sequentially, or simultaneously. In the first case, their forms are

$$\begin{aligned} (O_X^1, \perp, \perp)(A) &= \begin{array}{c|ccc} & l_1, \langle A_{l_1}, B_{l_1} \rangle & \dots & l_n, \langle A_{l_n}, B_{l_n} \rangle \\ \hline k_1, O_X^1(\langle A_{k_1}, B_{k_1} \rangle) & \langle M_{k_1, l_1}, N_{k_1, l_1} \rangle & \dots & \langle M_{k_1, l_n}, N_{k_1, l_n} \rangle \\ \vdots & \vdots & \ddots & \vdots \\ k_m, O_X^1(\langle A_{k_m}, B_{k_m} \rangle) & \langle M_{k_m, l_1}, N_{k_m, l_1} \rangle & \dots & \langle M_{k_m, l_n}, N_{k_m, l_n} \rangle \end{array}, \\ \\ (\perp, O_Y^2, \perp)(A) &= \begin{array}{c|ccc} & l_1, O_Y^2(\langle A_{l_1}, B_{l_1} \rangle) & \dots & l_n, O_Y^2(\langle A_{l_n}, B_{l_n} \rangle) \\ \hline k_1, \langle A_{k_1}, B_{k_1} \rangle & \langle M_{k_1, l_1}, N_{k_1, l_1} \rangle & \dots & \langle M_{k_1, l_n}, N_{k_1, l_n} \rangle \\ \vdots & \vdots & \ddots & \vdots \\ k_m, \langle A_{k_m}, B_{k_m} \rangle & \langle M_{k_m, l_1}, N_{k_m, l_1} \rangle & \dots & \langle M_{k_m, l_n}, N_{k_m, l_n} \rangle \end{array}, \end{aligned}$$

$$(\perp, \perp, O_Z^3)(A) = \begin{array}{c|ccc} & l_1, \langle A_{l_1}, B_{l_1} \rangle & \dots & l_n, \langle A_{l_n}, B_{l_n} \rangle \\ \hline k_1, \langle A_{k_1}, B_{k_1} \rangle & O_Z^3(\langle M_{k_1, l_1}, N_{k_1, l_1} \rangle) & \dots & O_Z^3(\langle M_{k_1, l_n}, N_{k_1, l_n} \rangle) \\ \vdots & \vdots & \ddots & \vdots \\ k_m, \langle A_{k_m}, B_{k_m} \rangle & O_Z^3(\langle M_{k_m, l_1}, N_{k_m, l_1} \rangle) & \dots & O_Z^3(\langle M_{k_m, l_n}, N_{k_m, l_n} \rangle) \end{array}$$

In the second case, the form of the triple of operators is

$$\begin{array}{c} (O_X^1, O_Y^2, O_Z^3)(A) \\ \\ = \begin{array}{c|ccc} & l_1, O_Y^2(\langle A_{l_1}, B_{l_1} \rangle) & \dots & l_n, O_Y^2(\langle A_{l_n}, B_{l_n} \rangle) \\ \hline k_1, O_X^1(\langle A_{k_1}, B_{k_1} \rangle) & O_Z^3(\langle M_{k_1, l_1}, N_{k_1, l_1} \rangle) & \dots & O_Z^3(\langle M_{k_1, l_n}, N_{k_1, l_n} \rangle) \\ \vdots & \vdots & \ddots & \vdots \\ k_i, O_X^1(\langle A_{k_i}, B_{k_i} \rangle) & O_Z^3(\langle M_{k_i, l_1}, N_{k_i, l_1} \rangle) & \dots & O_Z^3(\langle M_{k_i, l_n}, N_{k_i, l_n} \rangle) \\ \vdots & \vdots & \ddots & \vdots \\ k_m, O_X^1(\langle A_{k_m}, B_{k_m} \rangle) & O_Z^3(\langle M_{k_m, l_1}, N_{k_m, l_1} \rangle) & \dots & O_Z^3(\langle M_{k_m, l_n}, N_{k_m, l_n} \rangle) \end{array} \end{array}$$

The above can be applied directly to the other modal operators defined over IVIFSs described in [31].

Temporal and predicate IFIMs

6.1 Definition of a Temporal IFIM and Extended temporal IFIM

In [20] the definition of the object Temporal IFIM (TIFIM) was introduced on the basis of the paper [49] coauthored by Evdokia Sotirova, Veselina Bureva, Anthony Shannon and the author. However, here we will start with a simpler type of IMs – Temporal IMs (TIMs).

Let A be an IM which elements a_{k_i, l_j} are functions of the argument $\tau \in \mathcal{T}$, where $\mathcal{T}$ is a fixed temporal or time-scale. Therefore, the elements τ are time-moments. Now, the TIM in the fixed time-moment τ will have the form:

$$A(\tau) = [K, L, \{a_{k_i, l_j}(\tau)\}] = \begin{array}{c|cccc} & l_1 & \dots & l_j & \dots & l_n \\ \hline k_1 & a_{k_1, l_1}(\tau) & \dots & a_{k_1, l_j}(\tau) & \dots & a_{k_1, l_n}(\tau) \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_i & a_{k_i, l_1}(\tau) & \dots & a_{k_i, l_j}(\tau) & \dots & a_{k_i, l_n}(\tau) \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_m & a_{k_m, l_1}(\tau) & \dots & a_{k_m, l_j}(\tau) & \dots & a_{k_m, l_n}(\tau) \end{array}.$$

Let here and below $\mathcal{T}$ be a time-scale. Then, we obtain the following set of TIMs:

$$A(\mathcal{T}) = \{[K, L, \{a_{k_i, l_j}(\tau)\}] | \tau \in \mathcal{T}\}$$

$$= \left\{ \begin{array}{c|cccc} & l_1 & \dots & l_j & \dots & l_n \\ \hline k_1 & a_{k_1, l_1}(\tau) & \dots & a_{k_1, l_j}(\tau) & \dots & a_{k_1, l_n}(\tau) \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_i & a_{k_i, l_1}(\tau) & \dots & a_{k_i, l_j}(\tau) & \dots & a_{k_i, l_n}(\tau) \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_m & a_{k_m, l_1}(\tau) & \dots & a_{k_m, l_j}(\tau) & \dots & a_{k_m, l_n}(\tau) \end{array} \mid \tau \in \mathcal{T} \right\}.$$

Therefore, we can modify this TIM to the following 3-dimensional form

$$A(\mathcal{T}) = [K, L, \mathcal{T}, \{a_{k_i, l_j}(\tau)\}]$$

$$= \left\{ \begin{array}{c|cccc} \tau & l_1 & \dots & l_j & \dots & l_n \\ \hline k_1 & a_{k_1, l_1}(\tau) & \dots & a_{k_1, l_j}(\tau) & \dots & a_{k_1, l_n}(\tau) \\ \vdots & \vdots & & \vdots & & \vdots \\ k_i & a_{k_i, l_1}(\tau) & \dots & a_{k_i, l_j}(\tau) & \dots & a_{k_i, l_n}(\tau) \\ \vdots & \vdots & & \vdots & & \vdots \\ k_m & a_{k_m, l_1}(\tau) & \dots & a_{k_m, l_j}(\tau) & \dots & a_{k_m, l_n}(\tau) \end{array} \mid \tau \in \mathcal{T} \right\}.$$

The concept of a TIM can be extended in at least three directions.

First, we can change the IM indices k_i, l_j with functions of an argument τ and then we will obtain an Extended TIM (ETIM):

$$A(\mathcal{T}) = [K, L, \mathcal{T}, \{a_{k_i, l_j}(\tau)\}]$$

$$= \left\{ \begin{array}{c|cccc} \tau & l_1(\tau) & \dots & l_j(\tau) & \dots & l_n(\tau) \\ \hline k_1(\tau) & a_{k_1, l_1}(\tau) & \dots & a_{k_1, l_j}(\tau) & \dots & a_{k_1, l_n}(\tau) \\ \vdots & \vdots & & \vdots & & \vdots \\ k_i(\tau) & a_{k_i, l_1}(\tau) & \dots & a_{k_i, l_j}(\tau) & \dots & a_{k_i, l_n}(\tau) \\ \vdots & \vdots & & \vdots & & \vdots \\ k_m(\tau) & a_{k_m, l_1}(\tau) & \dots & a_{k_m, l_j}(\tau) & \dots & a_{k_m, l_n}(\tau) \end{array} \mid \tau \in \mathcal{T} \right\}.$$

Second, we can use as elements of A (again functions of τ) some n -dimensional vectors and in a result we will obtain a $(n + 1)$ -dimensional TIM.

The most interesting is the third direction of TIM extension – the Temporal Intuitionistic Fuzzy IM (TIFIM) that is defined by

$$A(\mathcal{T}) = [K, L, \mathcal{T}, \{\langle \mu_{k_i, l_j, \tau}, \nu_{k_i, l_j, \tau} \rangle\}]$$

$$\equiv \left\{ \begin{array}{c|cccc} & l_1 & \dots & l_n & \\ \hline k_1 & \langle \mu_{k_1, l_1, \tau}, \nu_{k_1, l_1, \tau} \rangle & \dots & \langle \mu_{k_1, l_n, \tau}, \nu_{k_1, l_n, \tau} \rangle & \\ \vdots & \vdots & & \vdots & \\ k_i & \langle \mu_{k_i, l_1, \tau}, \nu_{k_i, l_1, \tau} \rangle & \dots & \langle \mu_{k_i, l_n, \tau}, \nu_{k_i, l_n, \tau} \rangle & \\ \vdots & \vdots & & \vdots & \\ k_m & \langle \mu_{k_m, l_1, \tau}, \nu_{k_m, l_1, \tau} \rangle & \dots & \langle \mu_{k_m, l_n, \tau}, \nu_{k_m, l_n, \tau} \rangle & \end{array} \mid \tau \in \mathcal{T} \right\},$$

where for every $\tau \in \mathcal{T}$, $1 \leq i \leq m$, $1 \leq j \leq n$:

$$\mu_{k_i, l_j, \tau} = \mu_{k_i, l_j}(\tau) \in [0, 1],$$

$$\nu_{k_i, l_j, \tau} = \nu_{k_i, l_j}(\tau) \in [0, 1],$$

$$\mu_{k_i, l_j, \tau} + \nu_{k_i, l_j, \tau} \in [0, 1].$$

By analogy with the Sections 2.1 and 4.4, we extend the concept of a TIFIM, defining an Extended TIFIM (ETIFIM; see [25, 38]), by:

$$A^*(\mathcal{T}) = [K^*(\mathcal{T}), L^*(\mathcal{T}), \{\langle \mu_{k_i, l_j, \tau}, \nu_{k_i, l_j, \tau} \rangle\}]$$

$$\equiv \left\{ \begin{array}{c|ccc} & l_1, \langle \alpha_{1, \tau}^l, \beta_{1, \tau}^l \rangle & \dots & l_n, \langle \alpha_{n, \tau}^l, \beta_{n, \tau}^l \rangle \\ \hline k_1, \langle \alpha_{1, \tau}^k, \beta_{1, \tau}^k \rangle & \langle \mu_{k_1, l_1, \tau}, \nu_{k_1, l_1, \tau} \rangle & \dots & \langle \mu_{k_1, l_n, \tau}, \nu_{k_1, l_n, \tau} \rangle \\ \vdots & \vdots & \ddots & \vdots \\ k_i, \langle \alpha_{i, \tau}^k, \beta_{i, \tau}^k \rangle & \langle \mu_{k_i, l_1, \tau}, \nu_{k_i, l_1, \tau} \rangle & \dots & \langle \mu_{k_i, l_n, \tau}, \nu_{k_i, l_n, \tau} \rangle \\ \vdots & \vdots & \ddots & \vdots \\ k_m, \langle \alpha_{m, \tau}^k, \beta_{m, \tau}^k \rangle & \langle \mu_{k_m, l_1, \tau}, \nu_{k_m, l_1, \tau} \rangle & \dots & \langle \mu_{k_m, l_n, \tau}, \nu_{k_m, l_n, \tau} \rangle \end{array} \mid \tau \in \mathcal{T} \right\},$$

where for every $1 \leq i \leq m, 1 \leq j \leq n$ and for each $\tau \in \mathcal{T}$:

$$\mu_{k_i, l_j, \tau} = \mu_{k_i, l_j}(\tau), \quad \nu_{k_i, l_j, \tau} = \nu_{k_i, l_j}(\tau), \quad \mu_{k_i, l_j, \tau} + \nu_{k_i, l_j, \tau} \in [0, 1],$$

$$\alpha_{i, \tau}^k = \alpha_i^k(\tau), \quad \beta_{i, \tau}^k = \beta_i^k(\tau), \quad \alpha_{i, \tau}^k + \beta_{i, \tau}^k \in [0, 1],$$

$$\alpha_{j, \tau}^l = \alpha_j^l(\tau), \quad \beta_{j, \tau}^l = \beta_j^l(\tau), \quad \alpha_{j, \tau}^l + \beta_{j, \tau}^l \in [0, 1]$$

and

$$\begin{aligned} K^*(\mathcal{T}) &= \{\langle k_i, \alpha_{i, \tau}^k, \beta_{i, \tau}^k \rangle \mid k_i \in K \text{ \& } \tau \in \mathcal{T}\} \\ &= \{\langle k_i, \alpha_{i, \tau}^k, \beta_{i, \tau}^k \rangle \mid 1 \leq i \leq m \text{ \& } \tau \in \mathcal{T}\}, \end{aligned}$$

$$\begin{aligned} L^*(\mathcal{T}) &= \{\langle l_j, \alpha_{j, \tau}^l, \beta_{j, \tau}^l \rangle \mid l_j \in L \text{ \& } \tau \in \mathcal{T}\} \\ &= \{\langle l_j, \alpha_{j, \tau}^l, \beta_{j, \tau}^l \rangle \mid 1 \leq j \leq n \text{ \& } \tau \in \mathcal{T}\}. \end{aligned}$$

Let

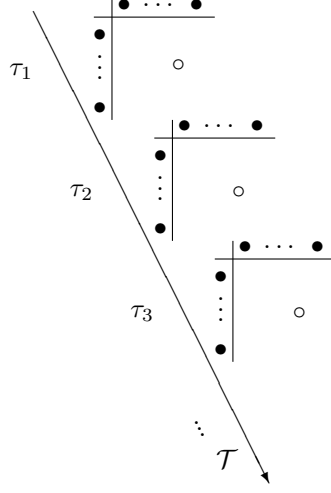
$$\begin{aligned} K^*(\mathcal{T}) \subset P^*(\mathcal{T}) \text{ iff } (K \subset P) \text{ \& } (\forall \tau \in \mathcal{T}), \\ (\forall k_i = p_i \in K : (\alpha_{i, \tau}^k < \alpha_{i, \tau}^p) \text{ \& } (\beta_{i, \tau}^k > \beta_{i, \tau}^p)), \end{aligned}$$

$$\begin{aligned} K^*(\mathcal{T}) \subseteq P^*(\mathcal{T}) \text{ iff } (K \subseteq P) \text{ \& } (\forall \tau \in \mathcal{T}) \\ (\forall k_i = p_i \in K : (\alpha_{i, \tau}^k \leq \alpha_{i, \tau}^p) \text{ \& } (\beta_{i, \tau}^k \geq \beta_{i, \tau}^p)), \end{aligned}$$

and similarly for set L^* .

We must mention that the new IM is 3-dimensional by analogy with the IM described in the beginning of this section. The indices for the third dimension of the TIFIM and ETIFIM are elements of a time-scale $\mathcal{T}$ (see Figure 6.1).

In the next section, we will describe the operations, relations and operators over the ETIFIMs.

Figure 6.1. ETIFIM as elements of a time-scale $\mathcal{T}$.

6.2 Operations over ETIFIMs

For the ETIFIMs

$$\begin{aligned} A^*(\mathcal{T}) &= [K^*(\mathcal{T}), L^*(\mathcal{T}), \{\langle \mu_{k_i, l_j, \tau}, \nu_{k_i, l_j, \tau} \rangle\}], \\ B^*(\mathcal{T}) &= [P^*(\mathcal{T}), Q^*(\mathcal{T}), \{\langle \rho_{p_r, q_s, \tau}, \sigma_{p_r, q_s, \tau} \rangle\}], \end{aligned}$$

and for $\circ = (\circ_1, \circ_2)$, $* = (*_2, *_2) \in \{(\max, \min), (\min, \max)\}$, the operations are the following (these definitions are extensions of the definitions from [25, 38]).

Addition

$$A^*(\mathcal{T}) \oplus_{(\circ, *)} B^*(\mathcal{T}) = [T^*(\mathcal{T}), V^*(\mathcal{T}), \{\langle \varphi_{t_u, v_w, \tau}, \psi_{t_u, v_w, \tau} \rangle\}],$$

where

$$\begin{aligned} T^*(\mathcal{T}) &= K^*(\mathcal{T}) \cup P^*(\mathcal{T}) = \{\langle t_u, \alpha_{u, \tau}^t, \beta_{u, \tau}^t \rangle | t_u \in K \cup P \text{ \& } \tau \in \mathcal{T}\}, \\ V^*(\mathcal{T}) &= L^*(\mathcal{T}) \cup Q^*(\mathcal{T}) = \{\langle v_w, \alpha_{w, \tau}^v, \beta_{w, \tau}^v \rangle | v_w \in L \cup Q \text{ \& } \tau \in \mathcal{T}\}, \end{aligned}$$

$$\alpha_{u, \tau}^t = \begin{cases} \alpha_{i, \tau}^k, & \text{if } t_u \in K - P \\ \alpha_{r, \tau}^p, & \text{if } t_u \in P - K, \\ \circ_1(\alpha_{i, \tau}^k, \alpha_{r, \tau}^p), & \text{if } t_u \in K \cap P \end{cases}$$

$$\beta_{w,\tau}^v = \begin{cases} \beta_{j,\tau}^l, & \text{if } v_w \in L - Q \\ \beta_{s,\tau}^q, & \text{if } t_w \in Q - L, \\ \circ_2(\beta_{j,\tau}^l, \beta_{s,\tau}^q), & \text{if } t_w \in L \cap Q \end{cases}$$

and

$$\begin{aligned} & \langle \varphi_{t_u, v_w, \tau}, \psi_{t_u, v_w, \tau} \rangle = \\ = & \begin{cases} \langle \mu_{k_i, l_j, \tau}, \nu_{k_i, l_j, \tau} \rangle, & \text{if } t_u = k_i \in K \text{ and } v_w = l_j \in L - Q \\ & \text{or } t_u = k_i \in K - P \text{ and } v_w = l_j \in L; \\ \langle \rho_{p_r, q_s, \tau}, \sigma_{p_r, q_s, \tau} \rangle, & \text{if } t_u = p_r \in P \text{ and } v_w = q_s \in Q - L \\ & \text{or } t_u = p_r \in P - K \text{ and } v_w = q_s \in Q; \\ \langle *1(\mu_{k_i, l_j, \tau}, \rho_{p_r, q_s, \tau}), & \text{if } t_u = k_i = p_r \in K \cap P \\ *2(\nu_{k_i, l_j, \tau}, \sigma_{p_r, q_s, \tau}) \rangle, & \text{and } v_w = l_j = q_s \in L \cap Q; \\ \langle 0, 1 \rangle, & \text{otherwise} \end{cases} \end{aligned}$$

Termwise multiplication

$$A^*(\mathcal{T}) \otimes_{(\circ, *)} B^*(\mathcal{T}) = [T^*(\mathcal{T}), V^*(\mathcal{T}), \{\langle \varphi_{t_u, v_w, \tau}, \psi_{t_u, v_w, \tau} \rangle\}],$$

where

$$\begin{aligned} T^*(\mathcal{T}) &= K^*(\mathcal{T}) \cap P^*(\mathcal{T}) = \{\langle t_u, \alpha_{u,\tau}^t, \beta_{u,\tau}^t \rangle \mid t_u \in K \cap P \text{ \& } \tau \in \mathcal{T}\}, \\ V^*(\mathcal{T}) &= L^*(\mathcal{T}) \cap Q^*(\mathcal{T}) = \{\langle v_w, \alpha_{w,\tau}^v, \beta_{w,\tau}^v \rangle \mid v_w \in L \cap Q \text{ \& } \tau \in \mathcal{T}\}, \\ \alpha_{u,\tau}^t &= \circ_1(\alpha_{i,\tau}^k, \alpha_{r,\tau}^p), \text{ for } t_u = k_i = p_r \in K \cap P, \\ \beta_{w,\tau}^v &= \circ_2(\beta_{j,\tau}^l, \beta_{s,\tau}^q), \text{ for } v_w = l_j = q_s \in L \cap Q \end{aligned}$$

and

$$\langle \varphi_{t_u, v_w, \tau}, \psi_{t_u, v_w, \tau} \rangle = \langle *1(\mu_{k_i, l_j, \tau}, \rho_{p_r, q_s, \tau}), *2(\nu_{k_i, l_j, \tau}, \sigma_{p_r, q_s, \tau}) \rangle.$$

Multiplication

$$A^*(\mathcal{T}) \odot_{(\circ, *)} B^*(\mathcal{T}) = [T^*(\mathcal{T}), V^*(\mathcal{T}), \{\langle \varphi_{t_u, v_w, \tau}, \psi_{t_u, v_w, \tau} \rangle\}],$$

where

$$\begin{aligned} T^*(\mathcal{T}) &= (K \cup (P - L))^*(\mathcal{T}) = \{\langle t_u, \alpha_{u,\tau}^t, \beta_{u,\tau}^t \rangle \mid t_u \in K \cup (P - L)\}, \\ V^*(\mathcal{T}) &= (Q \cup (L - P))^*(\mathcal{T}) = \{\langle v_w, \alpha_{w,\tau}^v, \beta_{w,\tau}^v \rangle \mid v_w \in Q \cup (L - P)\}, \\ \alpha_{u,\tau}^t &= \begin{cases} \alpha_{i,\tau}^k, & \text{if } t_u = k_i \in K \\ \alpha_{r,\tau}^p, & \text{if } t_u = p_r \in P - L \end{cases}, \end{aligned}$$

$$\beta_{w,\tau}^v = \begin{cases} \beta_{j,\tau}^l, & \text{if } v_w = l_j \in L - P \\ \beta_{s,\tau}^q, & \text{if } t_w = q_s \in Q \end{cases},$$

and

$$\begin{aligned} & \langle \varphi_{t_u, v_w, \tau}, \psi_{t_u, v_w, \tau} \rangle = \\ & = \begin{cases} \langle \mu_{k_i, l_j, \tau}, \nu_{k_i, l_j, \tau} \rangle, & \text{if } t_u = k_i \in K \\ & \text{and } v_w = l_j \in L - P - Q; \\ \langle \rho_{p_r, q_s, \tau}, \sigma_{p_r, q_s, \tau} \rangle, & \text{if } t_u = p_r \in P - L - K \\ & \text{and } v_w = q_s \in Q; \\ \langle \underset{l_j = p_r \in L \cap P}{\circ} (\min(\mu_{k_i, l_j, \tau}, \rho_{p_r, q_s, \tau})), & \text{if } t_u = k_i \in K \text{ and } v_w = q_s \in Q; \\ \underset{l_j = p_r \in L \cap P}{*} (\max(\nu_{k_i, l_j, \tau}, \sigma_{p_r, q_s, \tau})), & \\ \langle 0, 1 \rangle, & \text{otherwise} \end{cases} \end{aligned}$$

Structural subtraction

$$A^*(\mathcal{T}) \ominus B^*(\mathcal{T}) = [T^*(\mathcal{T}), V^*(\mathcal{T}), \{\langle \varphi_{t_u, v_w, \tau}, \psi_{t_u, v_w, \tau} \rangle\}],$$

where

$$\begin{aligned} T^*(\mathcal{T}) &= (K - P)^*(\mathcal{T}) = \{\langle t_u, \alpha_{u,\tau}^t, \beta_{u,\tau}^t \rangle | t_u \in K - P\}, \\ V^*(\mathcal{T}) &= (L - Q)^*(\mathcal{T}) = \{\langle v_w, \alpha_{w,\tau}^v, \beta_{w,\tau}^v \rangle | v_w \in L - Q\}, \end{aligned}$$

for the set-theoretic subtraction operation and

$$\begin{aligned} \alpha_{u,\tau}^t &= \alpha_{i,\tau}^k, \text{ for } t_u = k_i \in K - P, \\ \beta_{w,\tau}^v &= \beta_{j,\tau}^l, \text{ for } v_w = l_j \in L - Q \end{aligned}$$

and

$$\langle \varphi_{t_u, v_w, \tau}, \psi_{t_u, v_w, \tau} \rangle = \langle \mu_{k_i, l_j, \tau}, \nu_{k_i, l_j, \tau} \rangle$$

for $t_u = k_i \in K - P$ and $v_w = l_j \in L - Q$.

Negation

$$\neg A(\mathcal{T})^* = [T^*(\mathcal{T}), V^*(\mathcal{T}), \{\neg \langle \mu_{k_i, l_j, \tau}, \nu_{k_i, l_j, \tau} \rangle\}].$$

Operations “reduction”, “projection” and “substitution” coincide with the respective operations defined over IMs in [10], while the hierarchical operations over IMs are not applicable here.

6.3 Relations over ETIFIMs

Let the two ETIFIMs

$$\begin{aligned} A^*(\mathcal{T}) &= [K^*(\mathcal{T}), L^*(\mathcal{T}), \{\langle \mu_{k_i, l_j, \tau}, \nu_{k_i, l_j, \tau} \rangle\}], \\ B^*(\mathcal{T}) &= [P^*(\mathcal{T}), Q^*(\mathcal{T}), \{\langle \rho_{p_r, q_s, \tau}, \sigma_{p_r, q_s, \tau} \rangle\}] \end{aligned}$$

be given. We introduce the following definitions where $\subset$ and $\subseteq$ denote the relations “*strong inclusion*” and “*weak inclusion*”.

The strict relation “inclusion about matrix-dimension and elements” is

$$\begin{aligned} A^*(\mathcal{T}) \subset_d^e B^*(\mathcal{T}) \text{ iff } &(((K^*(\mathcal{T}) \subset P^*(\mathcal{T})) \ \& \ (L^*(\mathcal{T}) \subset Q^*(\mathcal{T}))) \\ &\vee ((K^*(\mathcal{T}) \subseteq P^*(\mathcal{T})) \ \& \ (L^*(\mathcal{T}) \subset Q^*(\mathcal{T}))) \\ &\vee (K^*(\mathcal{T}) \subset P^*(\mathcal{T})) \ \& \ (L^*(\mathcal{T}) \subseteq Q^*(\mathcal{T}))) \\ &\& \ (\forall \tau \in (\mathcal{T}))(\forall k \in K)(\forall l \in L)(\langle \mu_{k_i, l_j, \tau}, \nu_{k_i, l_j, \tau} \rangle = \langle \rho_{k_i, l_j, \tau}, \sigma_{k_i, l_j, \tau} \rangle). \end{aligned}$$

The non-strict relation “inclusion about matrix-dimension and elements” is

$$\begin{aligned} A^*(\mathcal{T}) \subseteq_d^e B^*(\mathcal{T}) \text{ iff } &(K^*(\mathcal{T}) \subseteq P^*(\mathcal{T})) \ \& \ (L^*(\mathcal{T}) \subseteq Q^*(\mathcal{T})) \\ &\& \ (\forall \tau \in (\mathcal{T}))(\forall k \in K)(\forall l \in L)(\langle \mu_{k_i, l_j, \tau}, \nu_{k_i, l_j, \tau} \rangle = \langle \rho_{k_i, l_j, \tau}, \sigma_{k_i, l_j, \tau} \rangle). \end{aligned}$$

The strict relation “inclusion about element values” is

$$\begin{aligned} A^*(\mathcal{T}) \subset_v^e B^*(\mathcal{T}) \text{ iff } &(K^*(\mathcal{T}) = P^*(\mathcal{T})) \ \& \ (L^*(\mathcal{T}) = Q^*(\mathcal{T})) \\ &\& \ (\forall \tau \in (\mathcal{T}))(\forall k \in K)(\forall l \in L)(\langle \mu_{k_i, l_j, \tau}, \nu_{k_i, l_j, \tau} \rangle < \langle \rho_{k_i, l_j, \tau}, \sigma_{k_i, l_j, \tau} \rangle). \end{aligned}$$

The non-strict relation “inclusion about element values” is

$$\begin{aligned} A^*(\mathcal{T}) \subseteq_v^e B^*(\mathcal{T}) \text{ iff } &(K^*(\mathcal{T}) = P^*(\mathcal{T})) \ \& \ (L^*(\mathcal{T}) = Q^*(\mathcal{T})) \\ &\& \ (\forall \tau \in (\mathcal{T}))(\forall k \in K)(\forall l \in L)(\langle \mu_{k_i, l_j, \tau}, \nu_{k_i, l_j, \tau} \rangle \leq \langle \rho_{k_i, l_j, \tau}, \sigma_{k_i, l_j, \tau} \rangle). \end{aligned}$$

The strict relation “inclusion about matrix-dimension and element values” is

$$\begin{aligned} A^*(\mathcal{T}) \subset^e B^*(\mathcal{T}) \text{ iff } &(((K^*(\mathcal{T}) \subset P^*(\mathcal{T})) \ \& \ (L^*(\mathcal{T}) \subset Q^*(\mathcal{T}))) \\ &\vee ((K^*(\mathcal{T}) \subseteq P^*(\mathcal{T})) \ \& \ (L^*(\mathcal{T}) \subset Q^*(\mathcal{T}))) \\ &\vee ((K^*(\mathcal{T}) \subset P^*(\mathcal{T})) \ \& \ (L^*(\mathcal{T}) \subseteq Q^*(\mathcal{T}))) \\ &\& \ (\forall \tau \in (\mathcal{T}))(\forall k \in K)(\forall l \in L)(\langle \mu_{k_i, l_j, \tau}, \nu_{k_i, l_j, \tau} \rangle < \langle \rho_{k_i, l_j, \tau}, \sigma_{k_i, l_j, \tau} \rangle). \end{aligned}$$

The non-strict relation “inclusion about matrix-dimension and element values” is

$$A^*(\mathcal{T}) \subseteq^e B^*(\mathcal{T}) \text{ iff } (K^*(\mathcal{T}) \subseteq P^*(\mathcal{T})) \ \& \ (L^*(\mathcal{T}) \subseteq Q^*(\mathcal{T})) \\ \& \ (\forall \tau \in (\mathcal{T}))(\forall k \in K)(\forall l \in L)(\langle \mu_{k_i, l_j, \tau}, \nu_{k_i, l_j, \tau} \rangle \leq \langle \rho_{k_i, l_j, \tau}, \sigma_{k_i, l_j, \tau} \rangle).$$

The strict relation “inclusion about matrix-dimension and indices” is

$$A^*(\mathcal{T}) \subset_d^i B^*(\mathcal{T}) \text{ iff } (((K^*(\mathcal{T}) \subset P^*(\mathcal{T})) \ \& \ (L^*(\mathcal{T}) \subset Q^*(\mathcal{T}))) \\ \vee ((K^*(\mathcal{T}) \subseteq P^*(\mathcal{T})) \ \& \ (L^*(\mathcal{T}) \subset Q^*(\mathcal{T}))) \\ \vee ((K^*(\mathcal{T}) \subset P^*(\mathcal{T})) \ \& \ (L^*(\mathcal{T}) \subseteq Q^*(\mathcal{T})))) \\ \& \ (\forall \tau \in (\mathcal{T}))(\forall k \in K)(\forall l \in L) \\ (\langle \alpha_{i, \tau}^k, \beta_{i, \tau}^k \rangle = \langle \alpha_{i, \tau}^p, \beta_{i, \tau}^p \rangle \ \& \ \langle \alpha_{i, \tau}^l, \beta_{i, \tau}^l \rangle = \langle \alpha_{i, \tau}^q, \beta_{i, \tau}^q \rangle).$$

The non-strict relation “inclusion about matrix-dimension and indices” is

$$A^*(\mathcal{T}) \subseteq_d^i B^*(\mathcal{T}) \text{ iff } (K^*(\mathcal{T}) \subseteq P^*(\mathcal{T})) \ \& \ (L^*(\mathcal{T}) \subseteq Q^*(\mathcal{T})) \\ \& \ (\forall \tau \in (\mathcal{T}))(\forall k \in K)(\forall l \in L) \\ (\langle \alpha_{i, \tau}^k, \beta_{i, \tau}^k \rangle = \langle \alpha_{i, \tau}^p, \beta_{i, \tau}^p \rangle \ \& \ \langle \alpha_{i, \tau}^l, \beta_{i, \tau}^l \rangle = \langle \alpha_{i, \tau}^q, \beta_{i, \tau}^q \rangle).$$

The strict relation “inclusion about matrix-dimension and index values” is

$$A^*(\mathcal{T}) \subset_v^i B^*(\mathcal{T}) \text{ iff } (K^*(\mathcal{T}) = P^*(\mathcal{T})) \ \& \ (L^*(\mathcal{T}) = Q^*(\mathcal{T})) \\ \& \ (\forall \tau \in (\mathcal{T}))(\forall k \in K)(\forall l \in L) \\ (\langle \alpha_{i, \tau}^k, \beta_{i, \tau}^k \rangle < \langle \alpha_{i, \tau}^p, \beta_{i, \tau}^p \rangle \ \& \ \langle \alpha_{i, \tau}^l, \beta_{i, \tau}^l \rangle < \langle \alpha_{i, \tau}^q, \beta_{i, \tau}^q \rangle).$$

The non-strict relation “inclusion about matrix-dimension and index values” is

$$A^*(\mathcal{T}) \subseteq_v^i B^*(\mathcal{T}) \text{ iff } (K^*(\mathcal{T}) = P^*(\mathcal{T})) \ \& \ (L^*(\mathcal{T}) = Q^*(\mathcal{T})) \\ \& \ (\forall \tau \in (\mathcal{T}))(\forall k \in K)(\forall l \in L) \\ (\langle \alpha_{i, \tau}^k, \beta_{i, \tau}^k \rangle \leq \langle \alpha_{i, \tau}^p, \beta_{i, \tau}^p \rangle \ \& \ \langle \alpha_{i, \tau}^l, \beta_{i, \tau}^l \rangle \leq \langle \alpha_{i, \tau}^q, \beta_{i, \tau}^q \rangle).$$

The strict relation “inclusion about index values” is

$$A^*(\mathcal{T}) \subset^i B^*(\mathcal{T}) \text{ iff } (((K^*(\mathcal{T}) \subset P^*(\mathcal{T})) \ \& \ (L^*(\mathcal{T}) \subset Q^*(\mathcal{T}))) \\ \vee ((K^*(\mathcal{T}) \subseteq P^*(\mathcal{T})) \ \& \ (L^*(\mathcal{T}) \subset Q^*(\mathcal{T}))) \\ \vee ((K^*(\mathcal{T}) \subset P^*(\mathcal{T})) \ \& \ (L^*(\mathcal{T}) \subseteq Q^*(\mathcal{T})))) \\ \& \ (\forall \tau \in (\mathcal{T}))(\forall k \in K)(\forall l \in L) \\ (\langle \alpha_{i, \tau}^k, \beta_{i, \tau}^k \rangle < \langle \alpha_{i, \tau}^p, \beta_{i, \tau}^p \rangle \ \& \ \langle \alpha_{i, \tau}^l, \beta_{i, \tau}^l \rangle < \langle \alpha_{i, \tau}^q, \beta_{i, \tau}^q \rangle).$$

The non-strict relation “inclusion about index values” is

$$\begin{aligned} A^*(\mathcal{T}) \subseteq^i B^*(\mathcal{T}) \text{ iff } & (K^*(\mathcal{T}) \subseteq P^*(\mathcal{T})) \ \& \ (L^*(\mathcal{T}) \subseteq Q^*(\mathcal{T})) \\ & \& \ (\forall \tau \in (\mathcal{T}))(\forall k \in K)(\forall l \in L) \\ & (\langle \alpha_{i,\tau}^k, \beta_{i,\tau}^k \rangle \leq \langle \alpha_{i,\tau}^p, \beta_{i,\tau}^p \rangle \ \& \ \langle \alpha_{i,\tau}^l, \beta_{i,\tau}^l \rangle \leq \langle \alpha_{i,\tau}^q, \beta_{i,\tau}^q \rangle). \end{aligned}$$

ETIFIM $A^*(\mathcal{T})$ has temporal strictly increasing elements, if

$$\begin{aligned} & (\forall \tau_1, \tau_2 \in \mathcal{T})(\tau_1 < \tau_2) \rightarrow (\forall k_i \in K)(\forall l_j \in L) \\ & (\langle \mu_{k_i, l_j, \tau_1}, \nu_{k_i, l_j, \tau_1} \rangle < \langle \rho_{k_i, l_j, \tau_2}, \sigma_{k_i, l_j, \tau_2} \rangle). \end{aligned}$$

ETIFIM $A^*(\mathcal{T})$ has temporal non-strictly increasing elements, if

$$\begin{aligned} & (\forall \tau_1, \tau_2 \in \mathcal{T})(\tau_1 < \tau_2) \rightarrow (\forall k_i \in K)(\forall l_j \in L) \\ & (\langle \mu_{k_i, l_j, \tau_1}, \nu_{k_i, l_j, \tau_1} \rangle \leq \langle \rho_{k_i, l_j, \tau_2}, \sigma_{k_i, l_j, \tau_2} \rangle). \end{aligned}$$

ETIFIM $A^*(\mathcal{T})$ has temporal strictly increasing indices, if

$$\begin{aligned} & (\forall \tau_1, \tau_2 \in \mathcal{T})(\tau_1 < \tau_2) \rightarrow (\forall k_i \in K)(\forall l_j \in L) \\ & (\langle \alpha_{i,\tau_1}^k, \beta_{i,\tau_1}^k \rangle < \langle \alpha_{i,\tau_2}^p, \beta_{i,\tau_2}^p \rangle \ \& \ \langle \alpha_{j,\tau_1}^l, \beta_{j,\tau_1}^l \rangle < \langle \alpha_{j,\tau_2}^q, \beta_{j,\tau_2}^q \rangle). \end{aligned}$$

ETIFIM $A^*(\mathcal{T})$ has temporal non-strictly increasing indices, if

$$\begin{aligned} & (\forall \tau_1, \tau_2 \in \mathcal{T})(\tau_1 < \tau_2) \rightarrow (\forall k_i \in K)(\forall l_j \in L) \\ & (\langle \alpha_{i,\tau_1}^k, \beta_{i,\tau_1}^k \rangle \leq \langle \alpha_{i,\tau_2}^p, \beta_{i,\tau_2}^p \rangle \ \& \ \langle \alpha_{j,\tau_1}^l, \beta_{j,\tau_1}^l \rangle \leq \langle \alpha_{j,\tau_2}^q, \beta_{j,\tau_2}^q \rangle). \end{aligned}$$

6.4 Specific operations over ETIFIMs

Let operations $\bar{\circ}$ and $\bar{*}$ are dual operations of operations $\circ$ and $*$, respectively. For example, pairs $(\circ, \bar{\circ})$ and $(*, \bar{*})$ can be any among pairs $(\max, \min)$, $(\min, \max)$. If $\circ$ and $*$ are average operations, then $\bar{\circ}$ and $\bar{*}$ are also average operations.

Let the time-scale $\mathcal{T}$ be fixed, let the ETIFIM

$$\begin{aligned} A^*(\mathcal{T}) &= [K^*(\mathcal{T}), L^*(\mathcal{T}), \{\langle \mu_{k_i, l_j, \tau}, \nu_{k_i, l_j, \tau} \rangle\}] \\ &\equiv \left\{ \begin{array}{c|ccc} & l_1, \langle \alpha_{1,\tau}^l, \beta_{1,\tau}^l \rangle & \dots & l_n, \langle \alpha_{n,\tau}^l, \beta_{n,\tau}^l \rangle \\ \hline k_1, \langle \alpha_{1,\tau}^k, \beta_{1,\tau}^k \rangle & \langle \mu_{k_1, l_1, \tau}, \nu_{k_1, l_1, \tau} \rangle & \dots & \langle \mu_{k_1, l_n, \tau}, \nu_{k_1, l_n, \tau} \rangle \\ \vdots & \vdots & \ddots & \vdots \\ k_m, \langle \alpha_{m,\tau}^k, \beta_{m,\tau}^k \rangle & \langle \mu_{k_m, l_1, \tau}, \nu_{k_m, l_1, \tau} \rangle & \dots & \langle \mu_{k_m, l_n, \tau}, \nu_{k_m, l_n, \tau} \rangle \end{array} \mid \tau \in \mathcal{T} \right\} \end{aligned}$$

be given and let $k_0 \notin K$ and $l_0 \notin L$ be two indices. Now, we introduce the following operations over it:

($\circ, *$)-row-aggregation

$$\rho_{(\circ, *)}(A^*(\mathcal{T}), k_0) = \left\{ \begin{array}{c|c} & l_1, \langle \alpha_{1,\tau}^l, \beta_{1,\tau}^l \rangle \quad \dots \\ \hline k_0, \langle \alpha_{1,\tau}^k, \beta_{1,\tau}^k \rangle & \langle \mu_{k_1,l_1,\tau}^*, \nu_{k_1,l_1,\tau}^{\bar{*}} \rangle \dots \\ \vdots & \vdots \\ k_m, \langle \alpha_{m,\tau}^k, \beta_{m,\tau}^k \rangle & \langle \mu_{k_m,l_m,\tau}^*, \nu_{k_m,l_m,\tau}^{\bar{*}} \rangle \dots \end{array} \mid \tau \in \mathcal{T} \right\},$$

($\circ, *$)-column-aggregation

$$\sigma_{(\circ, *)}(A^*(\mathcal{T}), l_0) = \left\{ \begin{array}{c|c} & l_0, \langle \alpha_{j,\tau}^l, \beta_{j,\tau}^l \rangle \\ \hline k_1, \langle \alpha_{1,\tau}^k, \beta_{1,\tau}^k \rangle & \langle \mu_{k_1,l_j,\tau}^*, \nu_{k_1,l_j,\tau}^{\bar{*}} \rangle \mid \tau \in \mathcal{T} \\ \vdots & \vdots \\ k_m, \langle \alpha_{m,\tau}^k, \beta_{m,\tau}^k \rangle & \langle \mu_{k_m,l_j,\tau}^*, \nu_{k_m,l_j,\tau}^{\bar{*}} \rangle \end{array} \right\}.$$

Let $\bullet$ and $\bar{\bullet}$ be two other dual operations, that may coincide with $(\circ, \bar{\circ})$ or $(*, \bar{*})$, or may be different. The following new aggregation operation can be defined:

($\circ, \bullet, *$)- $\mathcal{T}$ -aggregation

$$\rho_{(\circ, \bullet, *)}^{\mathcal{T}}(A^*(\mathcal{T})) = \left\{ \begin{array}{c|c} & l_1, \langle \alpha_{1,\tau}^l, \beta_{1,\tau}^l \rangle \quad \dots \\ \hline k_1, \langle \alpha_{1,\tau}^k, \beta_{1,\tau}^k \rangle & \langle \mu_{k_1,l_1,\tau}^*, \nu_{k_1,l_1,\tau}^{\bar{*}} \rangle \dots \\ \vdots & \vdots \\ k_m, \langle \alpha_{m,\tau}^k, \beta_{m,\tau}^k \rangle & \langle \mu_{k_m,l_1,\tau}^*, \nu_{k_m,l_1,\tau}^{\bar{*}} \rangle \dots \\ \vdots & \vdots \\ & \dots \quad l_n, \langle \alpha_{n,\tau}^l, \beta_{n,\tau}^l \rangle \\ \hline & \dots \quad \langle \mu_{k_1,l_n,\tau}^*, \nu_{k_1,l_n,\tau}^{\bar{*}} \rangle \\ & \vdots \\ & \dots \quad \langle \mu_{k_m,l_n,\tau}^*, \nu_{k_m,l_n,\tau}^{\bar{*}} \rangle \end{array} \right\}.$$

The above operations are extensions of the operations from IFIMs, EIFIMs and TIFIMs and have similar properties.

6.5 Definition of an n -dimensional temporal extended index matrix

Let $T = \{t_1, t_2, \dots, t_u\}$ be a fixed time-scale. An n -DTEIM with index sets $K_1, K_2, \dots, K_n$ ($K_1, K_2, \dots, K_n \subseteq \mathcal{I}^*$) and elements from set $\mathcal{X}$ that are functions of the elements of T , is called the object:

$$A(T) = [K_1, K_2, \dots, K_n, T, \{a_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}}(t)\}]$$

where $K_i = \{k_{i,1}, k_{i,2}, \dots, k_{i,m_i}\}$, $m_i \geq 1$ and $a_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}} \in \mathcal{X}$ for $1 \leq i \leq n$ and $1 \leq s_i \leq m_i$.

The n -DTEIM A has the following $\frac{n(n+1)}{2}$ different representations as 2-DTEIMs:

$$A = \left\{ \begin{array}{c|ccc} \langle k_{3,s_3}, \dots, k_{n,s_n} \rangle & k_{2,1} & \dots & k_{2,m_2} \\ \hline k_{1,1} & a_{k_{1,1}, k_{2,1}, k_{3,s_3}, \dots, k_{n,s_n}}(t) & \dots & a_{k_{1,1}, k_{2,m_2}, k_{3,s_3}, \dots, k_{n,s_n}}(t) \\ \vdots & \vdots & \ddots & \vdots \\ k_{1,i} & a_{k_{1,i}, k_{2,1}, k_{3,s_3}, \dots, k_{n,s_n}}(t) & \dots & a_{k_{1,i}, k_{2,m_2}, k_{3,s_3}, \dots, k_{n,s_n}}(t) \\ \vdots & \vdots & \ddots & \vdots \\ k_{1,m_1} & a_{k_{1,m_1}, k_{2,1}, k_{3,s_3}, \dots, k_{n,s_n}}(t) & \dots & a_{k_{1,m_1}, k_{2,m_2}, k_{3,s_3}, \dots, k_{n,s_n}}(t) \end{array} \right.$$

$$= \dots = \left\{ \begin{array}{c|ccc} \langle k_{1,s_1}, \dots, k_{n-2,s_{n-2}} \rangle & k_{n,1} & \dots & \\ \hline k_{n-1,1} & a_{k_{1,s_1}, \dots, k_{n-2,s_{n-2}}, k_{n-1,1}, k_{n,1}}(t) & \dots & \\ \vdots & \vdots & \ddots & \\ k_{n-1,i} & a_{k_{1,s_1}, \dots, k_{n-2,s_{n-2}}, k_{n-1,i}, k_{n,1}}(t) & \dots & \\ \vdots & \vdots & \ddots & \\ k_{n-1,m_{n-1}} & a_{k_{1,s_1}, \dots, k_{n-2,s_{n-2}}, k_{n-1,m_{n-1}}, k_{n,1}}(t) & \dots & \end{array} \right.$$

$$\begin{array}{c} \dots \\ \dots \\ \ddots \\ \dots \\ \ddots \\ \dots \end{array} \begin{array}{c} k_{n,m_n} \\ a_{k_{1,s_1}, \dots, k_{n-2,s_{n-2}}, k_{n-1,1}, k_{n,m_n}}(t) \\ \vdots \\ a_{k_{1,s_1}, \dots, k_{n-2,s_{n-2}}, k_{n-1,i}, k_{n,m_n}}(t) \\ \vdots \\ a_{k_{1,s_1}, \dots, k_{n-2,s_{n-2}}, k_{n-1,m_{n-1}}, k_{n,m_n}}(t) \end{array}$$

$$| \langle k_{1,s_1}, \dots, k_{n-2,s_{n-2}} \rangle \in K_1 \times \dots \times K_{n-2} \}.$$

Of course, the time-scale T can be one of the K -sets, but for the research above, it is important that it is a separate component.

6.6 Operations over n -Dimensional Temporal Extended Index Matrices

Here, we introduce the basic operations over n -DTEIMs, by analogy with [19, 20], but with the following important difference: in all previous research with TIMs, these matrices have elements related to *one* fixed temporal scale. Here, we will discuss the case, when each IM has its own temporal scale.

First, we mention that in Section 3.2 it is shown that there are $5(= 3! - 1)$ transpositions of a 3-DEIM. As we mentioned in Chapter 3, in the general case of n -DEIM, here — n DTEIM — there will exist $n! - 1$ transpositions.

Let us define

$$\begin{aligned} & [K_1, K_2, \dots, K_n, T, \{a_{k_1, s_1, k_2, s_2, \dots, k_n, s_n}(t)\}]^{[j_1, j_2, \dots, j_n]} \\ &= [K_{j_1}, K_{j_2}, \dots, K_{j_n}, T, \{a_{k_{j_1, s_{j_1}}, k_{j_2, s_{j_2}}, \dots, k_{j_n, s_{j_n}}}(t)\}]. \end{aligned}$$

Obviously, for the standard n -DEIM is valid

$$\begin{aligned} & [K_1, K_2, \dots, K_n, T, \{a_{k_1, s_1, k_2, s_2, \dots, k_n, s_n}(t)\}]^{[1, 2, \dots, n]} \\ &= [K_1, K_2, \dots, K_n, T, \{a_{k_1, s_1, k_2, s_2, \dots, k_n, s_n}(t)\}]. \end{aligned}$$

Second, let us have two n -DTEIMs

$$\begin{aligned} A &= [K_1, K_2, \dots, K_n, T_1, \{a_{k_1, s_1, k_2, s_2, \dots, k_n, s_n}(t_1)\}], \\ B &= [L_1, L_2, \dots, L_n, T_2, \{b_{l_1, r_1, l_2, r_2, \dots, l_n, r_n}(t_2)\}]. \end{aligned}$$

The case, when T_1 and T_2 coincide is the standard case that was discussed in [20] for 2DTEIMs. Here, we will assume that $T_1 \neq T_2$, but the case $T_1 \cap T_2 \neq \emptyset$ is possible.

Let everywhere below i be a natural number for which $1 \leq i \leq n$. Following [20, 29, 37, 54] we introduce the following operations over A and B . Firstly, we will describe the simpler case and, after this, the more complex one.

Operation “Addition”

$$A(T_1) \oplus_{(\circ)} B(T_2) = [K_1 \cup L_1, K_2 \cup L_2, \dots, K_n \cup L_n, T_1 \cup T_2, \{c_{p_1, q_1, p_2, q_2, \dots, p_n, q_n}(t)\}],$$

where

$$\begin{aligned}
& c_{p_1, q_1, p_2, q_2, \dots, p_n, q_n}(t) \\
= & \begin{cases} a_{k_1, s_1, k_2, s_2, \dots, k_n, s_n}(t_1), & \text{if for each } i: p_{i, q_i} = k_{i, s_i} \in K_i, \\ & \text{and } t = t_1 \in T_1 - T_2, \\ & \text{or if } t = t_1 \in T_1 \cup T_2, \text{ for almost all } i: \\ & p_{i, q_i} = k_{i, s_i} \in K_i \text{ and there exists } i \text{ such that} \\ & k_{i, s_i} \notin L_i \\ b_{l_1, r_1, l_2, r_2, \dots, l_n, r_n}(t_2), & \text{if for each } i: p_{i, q_i} = l_{i, r_i} \in L_i \\ & \text{and } t = t_2 \in T_2 - T_1, \\ & \text{or if } t = t_2 \in T_1 \cup T_2, \text{ for almost all } i: \\ & p_{i, q_i} = l_{i, r_i} \in L_i \text{ and there exists } i \text{ such that} \\ & l_{i, r_i} \notin K_i \\ a_{k_1, s_1, k_2, s_2, \dots, k_n, s_n}(t_1) & \text{if for each } i: p_{i, q_i} = k_{i, s_i} = l_{i, r_i} \in K_i \cap L_i \\ \circ b_{l_1, r_1, l_2, r_2, \dots, l_n, r_n}(t_2), & \text{and } t = t_1 = t_2 \in T_1 \cap T_2 \\ \perp, & \text{otherwise.} \end{cases}
\end{aligned}$$

Operation “Termwise multiplication”

$$\begin{aligned}
& A(T_1) \otimes_{(\circ)} B(T_2) \\
& = [K_1 \cap L_1, K_2 \cap L_2, \dots, K_n \cap L_n, T_1 \cap T_2, \{c_{p_1, q_1, p_2, q_2, \dots, p_n, q_n}(t)\}],
\end{aligned}$$

where

$$c_{p_1, q_1, p_2, q_2, \dots, p_n, q_n}(t) = a_{k_1, s_1, k_2, s_2, \dots, k_n, s_n}(t_1) \circ b_{l_1, r_1, l_2, r_2, \dots, l_n, r_n}(t_2)$$

for each i , $p_{i, q_i} = k_{i, s_i} = l_{i, r_i} \in K_i \cap L_i$ and for $t = t_1 = t_2 \in T_1 \cap T_2$.

Operations “Multiplication”

In this section, by analogy with Section 3.2, initially we discuss the case $n = 3$ and after this, the general case.

Standard operation “Multiplication”

Let $[x, y, z]$ be a permutation of the triple $[1, 2, 3]$.

Now, we add the temporal component, so the new IM will be 4DEIM, or the above sense, 3DTEIMs. Let the two such IMs:

$$A(T_1) = [K, L, H, T_1, \{a_{k_i, l_j, h_g}(t_1)\}]$$

and

$$B(T_2) = [P, Q, R, T_2, \{b_{p_r, q_s, e_d}(t_2)\}]$$

using different temporal scales be given, where $T_1 \neq T_2$, but it is possible that $T_1 \cap T_2 \neq \emptyset$.

Operation $[x, y, z]$ -multiplication is defined by:

$$A(T_1) \odot_{(o,*)}^{[x,y,z]} B(T_2) = A(T_1) \odot_{(o,*)} B^{[x,y,z]}(T_2),$$

where the first multiplication operation $\odot_{(o,*)}^{[x,y,z]}$ is:

$$A(T_1) \odot_{(o,*)} B(T_2) = [K \cup (P - L), Q \cup (L - P), H \cup R, T_1 \cup T_2, \{c_{t_u, v_w, x_y}(t)\}],$$

where

$$= \begin{cases} c_{t_u, v_w, x_y}(t) \\ a_{k_i, l_j, h_g}(t_1), & \text{if } t = t_1 \in T_1 \text{ and } t_u = k_i \in K \\ & \text{and } v_w = l_j \in L - P - Q \text{ and } x_y = h_g \in H \\ & \text{or } t_u = k_i \in K - P - Q \\ & \text{and } v_w = l_j \in L \text{ and } x_y = h_g \in H \\ & \text{or } t = t_1 \in T_1 - T_2; \\ b_{p_r, q_s, e_d}(t_2), & \text{if } t = t_2 \in T_2 \text{ and } t_u = p_r \in P \\ & \text{and } v_w = q_s \in Q - L - K \text{ and } x_y = e_d \in R \\ & \text{or } t_u = p_r \in P \\ & \text{and } v_w = q_s \in Q - L - K \text{ and } x_y = e_d \in R \\ & \text{or } t = t_2 \in T_2 - T_1; \\ \overset{\circ}{\underset{\in L \cap P}{l_j = p_r}}(a_{k_i, l_j, h_g}(t_1) * b_{p_r, q_s, e_d}(t_2)), & \text{if } t_u = k_i \in K \text{ and } v_w = q_s \in Q \\ & \text{and } x_y = h_g = e_d \in H \cap R \\ & \text{and } t = t_1 = t_2 \in T_1 \cap T_2; \\ \perp, & \text{otherwise.} \end{cases}$$

Now, we introduce two other operations “multiplication” and a new form of the above operation for the case, when A and B are 3DEIM in the forms from Section 3 for $n = 3$.

6.6.1 First new operation “multiplication”

$$A(T_1) \odot_{(o,*,1)} B(T_2) = [K_1, L_3, T_1 \cup T_2, \{c_{p_1, q_1, p_2, q_2}(t)\}],$$

where

$$= \begin{cases} c_{p_1, q_1, p_2, q_2}(t) = \\ \circ (a_{k_1, s_1, k_2, s_2, k_3, s_3} * b_{l_1, t_1, l_2, t_2, l_3, t_3}), & \text{if } p_1, q_1 = k_1, s_1 \in K_1 \\ & \text{and } p_2, q_2 = l_3, t_3 \in L_3 \\ & \text{and } t = t_1 = t_2 \in T_1 \cap T_2; \\ \perp, & \text{otherwise.} \end{cases}$$

Therefore, this operation generates a 2DTEIM, i.e., decreases the number of dimensions.

The number 1 in the record $\odot_{(\circ,*,1,1)}$ means that the first component of the first 3DEIM and the third (i.e., the first backwards) component of the second 3DEIM are used as the two components of the new, already, 2DEIM. These two components are interpreted as the first and last components of the new 3DEIM.

6.6.2 Second new operation “multiplication”

$$A(T_1) \odot_{(\circ,*,1,1)} B(T_2) = [K_1 \cup (L_2 - K_2), K_3 \cup L_1, L_3 \cup (K_2 - L_2), T_1 \cup T_2,$$

$$\{c_{p_{1,q_1}, p_{2,q_2}, p_{3,q_3}}(t)\}],$$

where

$$c_{p_{1,q_1}, p_{2,q_2}, p_{3,q_3}}(t) =$$

$$= \left\{ \begin{array}{ll} a_{k_{1,s_1}, k_{2,s_2}, k_{3,s_3}}(t_1), & \begin{array}{l} \text{if } t = t_1 \in T_1 \text{ and } p_{1,q_1} = k_{1,s_1} \in K_1 \\ \text{and } p_{2,q_2} = k_{2,s_2} \in K_2 - L_1 - L_2 \\ \text{and } p_{3,q_3} = k_{3,s_3} \in K_3, \\ \text{or } p_{1,q_1} = k_{1,s_1} \in K_1 - L_1 - L_2 \\ \text{and } p_{2,q_2} = k_{2,s_2} \in K_2 \\ \text{and } p_{3,q_3} = k_{3,s_3} \in K_3; \\ \text{or } t = t_1 \in T_1 - T_2 \end{array} \\ \\ b_{l_{1,r_1}, l_{2,r_2}, l_{3,r_3}}(t_2), & \begin{array}{l} \text{if } t = t_2 \in T_2 \\ \text{and } p_{1,q_1} = l_{1,r_1} \in L_1 \\ \text{and } p_{2,q_2} = l_{2,r_2} \in L_2 - K_1 - K_2 \\ \text{and } p_{3,q_3} = l_{3,r_3} \in L_3, \\ \text{or } p_{1,q_1} = l_{1,r_1} \in L_1 \\ \text{and } p_{2,q_2} = l_{2,r_2} \in L_2 - K_1 - K_2 \\ \text{and } p_{3,q_3} = l_{3,r_3} \in L_3; \\ \text{or } t = t_2 \in T_2 - T_1 \end{array} \\ \\ \underset{\substack{\circ \\ p_{2,q_2} = l_{1,t_1} \\ \in K_2 \cap L_1}}{a_{k_{1,s_1}, k_{2,s_2}, k_{3,s_3}}(t_1) * b_{l_{1,r_1}, l_{2,r_2}, l_{3,r_3}}(t_2)}, & \begin{array}{l} \text{if } p_{1,q_1} = k_{1,s_1} \in K_1, p_{2,q_2} = l_{2,r_2} \in L_2, \\ p_{3,q_3} = k_{3,s_3} = l_{3,r_3} \in K_3 \cap L_3 \\ \text{and } t = t_1 = t_2 \in T_1 \cap T_2 \end{array} \\ \\ \perp, & \text{otherwise} \end{array} \right.$$

This operation can be represented by the standard multiplication, because

$$\begin{aligned}
& A \odot_{(\circ, *, 1, 1)} B \\
&= [K_1 \cup (L_2 - K_2), K_3 \cup L_1, L_3 \cup (K_2 - L_2), T_1 \cup T_2, \{c_{p_1, q_1, p_2, q_2, p_3, q_3}(t)\}] \\
&= [K_1 \cup (L_2 - K_2), L_3 \cup (K_2 - L_2), K_3 \cup L_1, T_1 \cup T_2, \{c_{p_1, q_1, p_3, q_3, p_2, q_2}(t)\}]^{[1, 3, 2]} \\
&= ([K_1, K_2, K_3, T_1, \{a_{k_1, s_1, k_2, s_2, k_3, s_3}(t_1)\}]) \\
&\odot_{(\circ, *)} [L_2, L_3, L_1, T_2, \{b_{l_2, r_2, l_3, r_3, l_1, r_1}(t_2)\}]^{[1, 3, 2]} \\
&= ([K_1, K_2, K_3, T_1, \{a_{k_1, s_1, k_2, s_2, k_3, s_3}(t_1)\}]) \\
&\odot_{(\circ, *)} [L_1, L_2, L_3, T_2, \{b_{l_1, r_1, l_2, r_2, l_3, r_3}(t_2)\}]^{[3, 1, 2]}^{[1, 3, 2]} \\
&= ([K_1, K_2, K_3, T_1, \{a_{k_1, s_1, k_2, s_2, k_3, s_3}(t_1)\}]) \\
&\odot_{(\circ, *)}^{[3, 1, 2]} [L_1, L_2, L_3, T_2, \{b_{l_1, r_1, l_2, r_2, l_3, r_3}(t_2)\}]^{[1, 3, 2]}.
\end{aligned}$$

Therefore, this operation generates a 3DTEIM, i.e., preserves the number of dimensions.

The partial case when index sets $K_2 = L_2$, is very interesting. In it we obtain

$$A(T_1) \odot_{(\circ, *, 1, 1)} B(T_2) = [K_1, K_3 \cup L_1, L_3, T_1 \cup T_2, \{c_{p_1, q_1, p_2, q_2, p_3, q_3}(t)\}],$$

where

$$\begin{aligned}
& c_{p_1, q_1, p_2, q_2, p_3, q_3}(t) = \\
&= \begin{cases} \overset{\circ}{p_2, q_2 = l_1, t_1}_{\in K_2 \cap L_1} (a_{k_1, s_1, k_2, s_2, k_3, s_3}(t_1) * b_{l_1, r_1, l_2, r_2, l_3, r_3}(t_2)), & \text{if } p_1, q_1 = k_1, s_1 \in K_1 \text{ and } p_2, q_2 = l_2, t_2 \in L_2 \\ & \text{and } p_3, q_3 = k_3, s_3 = l_3, r_3 \in K_3 \cap L_3 \\ & \text{and } t = t_1 = t_2 \in T_1 \cup T_2; \\ \perp, & \text{otherwise.} \end{cases}
\end{aligned}$$

Exactly this form of the second operation corresponds to the record $\odot_{(\circ, *, 1, 1)}$. Here, the first 1 has the above sense, while the second 1 means that one component from the first and of the second 3DTEIM (the third of the first 3DEIM and the first of the second 3DTEIM) will be used as a second component of the resulting 3DTEIM.

6.6.3 Third new operation “multiplication”

$$A(T_1) \odot_{(\circ, *, 2)} B(T_1) = [K_1, K_2, L_3, L_4, T_1 \cup T_2, \{c_{p_1, q_1, p_2, q_2, p_3, q_3, p_4, q_4}(t)\}].$$

where

$$= \begin{cases} c_{p_1, q_1, p_2, q_2, p_3, q_3, p_4, q_4}(t) \\ \circ_{\substack{k_3, s_3 = l_1, r_1 \\ \in K_3 \cap L_1}} (a_{k_1, s_1, k_2, s_2, k_3, s_3}(t_1) * b_{l_1, r_1, l_2, r_2, l_3, r_3}(r_2)), & \begin{aligned} &\text{if } p_{1, q_1} = k_{1, s_1} \in K_1 \\ &\text{and } p_{2, q_2} = k_{2, s_2} \in K_2 \\ &\text{and } p_{3, q_3} = l_{2, r_2} \in L_2 \\ &\text{and } p_{4, q_4} = l_{3, r_3} \in L_3 \\ &\text{and } t = t_1 = t_2 \in T_1 \cap T_2; \end{aligned} \\ \perp, & \text{otherwise.} \end{cases}$$

Therefore, this operation generates a 4-DTEIM, i.e., increases the number of dimensions.

The number 2 in the record $\odot_{(\circ, *, 2)}$ means that the first two components of the first 3-DTEIM and the second two (i.e., the first two backwards) components of the second 3-DTEIM are used as the four components of the new already 4-DTEIM.

Following the sense of the introduced notation, we can define the general case of the operation multiplication.

General case of the operation “Multiplication”

Let $A(T_1)$ and $B(T_2)$ be the two n -DTEIM from Section 3. Then, we define one of the possible general cases for operation “multiplication”. It includes the first, third and the partial case of the second operations introduced above. By the moment, an **Open Problem** is what is the general form for the second multiplication.

Let α and β be two natural numbers for which $1 \leq \alpha \leq n-1$, $0 \leq \beta \leq n-2$ and $\alpha + \beta \leq n-1$. Then, we obtain the following $(2\alpha + \beta)$ -DEIM:

$$A(T_1) \odot_{(\circ, *, \alpha, \beta)} B(T_2) = [K_1, \dots, K_\alpha, K_{n-\beta+1} \cup L_1, \dots, K_n \cup L_\beta, \\ L_{n-\alpha+1}, \dots, L_n, T_1 \cup T_2, \{c_{p_1, q_1, p_2, q_2, \dots, p_{2\alpha+\beta}, q_{2\alpha+\beta}}(t)\}],$$

where

$$= \left\{ \begin{array}{ll} c_{p_{1,q_1}, p_{2,q_2}, \dots, p_{2\alpha+\beta}, q_{2\alpha+\beta}}(t) & \\ \circ \quad (ak_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}(t_1)) & \text{if } p_{1,q_1} = k_{1,s_1} \in K_1 \\ k_{\alpha+1, s_{\alpha+1}} = l_{\beta+1, t_{\beta+1}} \in K_{\alpha+1} \cap L_{\beta+1} & \dots \\ \dots & p_{\alpha, q_{\alpha}} = k_{\alpha, s_{\alpha}} \in K_{\alpha} \\ k_{n-\beta, s_{n-\beta}} = l_{n-\alpha, t_{n-\alpha}} \in K_{n-\beta} \cap L_{n-\alpha} & p_{\alpha+1, q_{\alpha+1}} = k_{n-\beta+1, s_{n-\beta+1}} \\ & = l_{1, t_1} \in K_{n-\beta+1} \cap L_1 \\ & \dots \\ & p_{\alpha+\beta, q_{\alpha+\beta}} = k_{n, s_n} \\ & = l_{\beta, r_{\beta}} \in K_n \cap L_{\beta} \\ & p_{\alpha+\beta+1, q_{\alpha+\beta+1}} \\ & = l_{n-\alpha+1, r_{n-\alpha+1}} \in L_{n-\alpha+1} \\ & \dots \\ & p_{2\alpha+\beta, q_{2\alpha+\beta}} = l_{n, t_n} \in L_n \\ & \text{and } t = t_1 = t_2 \in T_1 \cap T_2 \\ & \perp, \quad \text{otherwise} \end{array} \right.$$

Operation “Structural subtraction”

$$A(T_1) \ominus B(T_2) = [K_1 - L_1, K_2 - L_2, \dots, K_n - L_n, T_1 - T_2, \{c_{p_{1,q_1}, p_{2,q_2}, \dots, p_{n,q_n}}(t_1)\}],$$

where $c_{p_{1,q_1}, p_{2,q_2}, \dots, p_{n,q_n}}(t_1) = a_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}}(t_1)$ for each i , $p_{i,q_i} = k_{i,s_i} \in K_i - L_i$ and $t_1 \in T_1$.

Operations $\circ$ with a constant

$$\alpha \circ A(T) = [K_1, K_2, \dots, K_n, T, \{\alpha \circ a_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}}(t)\}],$$

where α is a (real) constant.

Now, this operation can be extended (for the first time) to the form

$$\alpha(T).A(T) = [K_1, K_2, \dots, K_n, T, \{\alpha(t)a_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}}(t)\}],$$

where $\alpha(T)$ is a function of T and $\alpha(t)$ is its value at time-moment $t \in T$.

Operations “Projection”

Let $L_i \subseteq K_i$. Then,

$$\text{pr}_{L_1, L_2, \dots, L_n} A(T) = [L_1, L_2, \dots, L_n, T, \{b_{l_{1,r_1}, l_{2,r_2}, \dots, l_{n,r_n}}(t)\}],$$

where $b_{l_{1,r_1}, l_{2,r_2}, \dots, l_{n,r_n}}(t) = a_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}}(t)$ for each i , $l_{i,r_i} = k_{i,s_i} \in L_i$ and $t \in T$.

Now, this operation can be modified (for the first time) to the form

$$\text{pr}_{L_1, L_2, \dots, L_n, T'} A(T) = [L_1, L_2, \dots, L_n, T', \{b_{l_{1,r_1}, l_{2,r_2}, \dots, l_{n,r_n}}(t)\}],$$

where $b_{l_{1,r_1}, l_{2,r_2}, \dots, l_{n,r_n}}(t) = a_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}}(t)$ for each i , $l_{i,r_i} = k_{i,s_i} \in L_i$ and $t \in T' \subseteq T$.

Operations “Reduction”

First, we introduce operation $(\perp, \dots, \perp, k_{i,r_i}, \perp, \dots, \perp)$ -reduction of a given n -DTEIM $A(T)$ by

$$A_{(\perp, \dots, \perp, k_{i,r_i}, \perp, \dots, \perp)}(T) = [K_1, \dots, K_{i-1}, K_i - \{k_{i,r_i}\}, K_{i+1}, \dots, K_n, T, \{c_{p_1,q_1}, p_2, q_2, \dots, p_n, q_n(t)\}],$$

where $c_{p_1,q_1}, p_2, q_2, \dots, p_n, q_n(t) = a_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}}(t)$ and $p_j, q_j = k_{j,s_j} \in K_j$ for $1 \leq j \leq n$ and $j \neq i$ and $t \in T$.

Second, we define

$$\begin{aligned} & A_{(k_{1,r_1}, k_{2,r_2}, \dots, k_{n,r_n})}(T) \\ &= (\dots (A_{(k_{1,r_1}, \perp, \dots, \perp)}(T))_{(\perp, k_{2,r_2}, \perp, \dots, \perp)} \dots)_{(\perp, \dots, \perp, k_{n,r_n})} \\ &= [K_1 - \{k_{1,r_1}\}, K_2 - \{k_{2,r_2}\}, \dots, K_n - \{k_{n,r_n}\}, T, \{c_{p_1,q_1}, p_2, q_2, \dots, p_n, q_n(t)\}], \end{aligned}$$

where $c_{p_1,q_1}, p_2, q_2, \dots, p_n, q_n(t) = a_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}}(t)$ and for each i : $p_{i,q_i} = k_{i,s_i} \in K_i - \{k_{i,r_i}\}$ and $1 \leq p_{i,q_i} \leq m_i$.

Third, let for each i : $L_i = \{l_{i,q_1}, l_{i,q_2}, \dots, l_{i,q_{r_i}}\} \subseteq K_i$, where $1 \leq q_{r_i} \leq m_i$. Now, we define the following operations:

$$\begin{aligned} & A_{(\perp, \dots, \perp, L_i, \perp, \dots, \perp)}(T) \\ &= (\dots ((A_{(\perp, \dots, \perp, l_{i,q_1}, \perp, \dots, \perp)}(T))_{(\perp, \dots, \perp, l_{i,q_2}, \perp, \dots, \perp)} \dots)_{(\perp, \dots, \perp, l_{i,q_{r_i}}, \perp, \dots, \perp)}), \\ & A_{(L_1, L_2, \dots, L_n)}(T) = (\dots ((A_{(L_1, \perp, \dots, \perp)}(T))_{(\perp, L_2, \perp, \dots, \perp)} \dots)_{(\perp, \dots, \perp, L_n)}). \end{aligned}$$

Operations “Substitution”

Let the n -DTEIM $A(T)$ be given.

First, local substitution over $A(T)$ is defined for the pairs of indices (l_{1,p_1}, k_{1,p_1}) and/or (l_{2,p_2}, k_{2,p_2}) and/or ... and/or (l_{n,p_n}, k_{n,p_n}) , respectively, for arbitrary indices p_i ($1 \leq p_i \leq m_i$), by

$$\begin{aligned} & \left[\frac{l_{1,p_1}}{k_{1,p_1}}; \perp; \dots; \perp \right] A(T) \\ &= [(K_1 - \{k_{1,p_1}\}) \cup \{l_{1,p_1}\}, K_2, \dots, K_n, T, \{a_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}}(t)\}] \\ & \left[\perp; \frac{l_{2,p_2}}{k_{2,p_2}}; \perp; \dots; \perp \right] A(T) \\ &= [K_1, (K_2 - \{k_{2,p_2}\}) \cup \{l_{2,p_2}\}, K_3, \dots, K_n, T, \{a_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}}(t)\}] \\ & \left[\perp; \dots; \perp; \frac{l_{n,p_n}}{k_{n,p_n}} \right] A(T) \\ &= [K_1, K_2, \dots, K_{n-1}, (K_n - \{k_{n,p_n}\}) \cup \{l_{n,p_n}\}, T, \{a_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}}(t)\}]. \end{aligned}$$

Second,

$$\begin{aligned}
& \left[\frac{l_{1,p_1}}{k_{1,p_1}}; \frac{l_{2,p_2}}{k_{2,p_2}}; \dots; \frac{l_{n,p_n}}{k_{n,p_n}} \right] A(T) \\
&= \left[\frac{l_1}{k_1}; \perp; \dots; \perp \right] \left[\perp; \frac{l_2}{k_2}; \perp; \dots; \perp \right] \dots \left[\perp; \perp; \dots; \perp; \frac{l_n}{k_n} \right] A(T) \\
&= [(K_1 - \{k_{1,p_1}\}) \cup \{l_{1,p_1}\}, (K_2 - \{k_{2,p_2}\}) \cup \{l_{2,p_2}\}, \dots, \\
&\quad (K_n - \{k_{n,p_n}\}) \cup \{l_{n,p_n}\}, T, \{a_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}}(t)\}].
\end{aligned}$$

Let the sets of indices $L_1, L_2, \dots, L_n$ be given, where $\text{card}(L_i) = \text{card}(K_i)$.

Third, for them we define sequentially:

$$\begin{aligned}
& \left[\perp; \dots; \perp; \frac{L_i}{K_i}; \perp; \dots; \perp \right] A(T) \\
&= \left[\perp; \dots; \perp; \frac{l_{i,1}}{k_{i,1}} \frac{l_{i,2}}{k_{i,2}} \dots \frac{l_{i,m_i}}{k_{i,m_i}}; \perp; \perp \right] A(T), \\
& \left[\frac{L_1}{K_1}; \frac{L_2}{K_2}; \dots; \frac{L_n}{K_n} \right] A(T) \\
&= \left[\frac{L_1}{K_1}; \perp; \dots; \perp \right] \left[\perp; \frac{L_2}{K_2}; \perp; \dots; \perp \right] \dots \left[\perp; \dots; \perp; \frac{L_n}{K_n} \right] A(T).
\end{aligned}$$

6.7 Other operations over n -DEIMs over different temporal scales

Let us define for the temporal scales T' and T'' and for $t'' \in T''$:

$$\begin{aligned}
\tau_{\max}(t'', T') &= \max\{t \mid t \in T' \ \& \ t \leq t''\}, \\
\tau_{\min}(t'', T') &= \min\{t \mid t \in T' \ \& \ t \geq t''\}.
\end{aligned}$$

Now, we re-write a part of the above definitions of operations, following their order above and changing only parts of their components.

Let $\# \in \{\tau_{\min}, \tau_{\max}\}$.

Operation “Addition”

$$\begin{aligned}
& A(T_1) \oplus_{(\circ)}^{[*]} B(T_2) \\
&= [K_1 \cup L_1, K_2 \cup L_2, \dots, K_n \cup L_n, T_1 \cup T_2, \{c_{p_1, q_1, p_2, q_2, \dots, p_n, q_n}(t)\}],
\end{aligned}$$

where

$$c_{p_1, q_1, p_2, q_2, \dots, p_n, q_n}(t) = \left\{ \begin{array}{ll} a_{k_1, s_1, k_2, s_2, \dots, k_n, s_n}(t_1), & \begin{array}{l} \text{if for each } i: p_{i, q_i} = k_{i, s_i} \in K_i, \\ \text{and } t = t_1 \in T_1 - T_2, \\ \text{or if } t = t_1 \in T_1 \cup T_2, \text{ for almost all } i: \\ p_{i, q_i} = k_{i, s_i} \in K_i \\ \text{and there is } i \text{ such that } k_{i, s_i} \notin L_i; \end{array} \\ b_{l_1, r_1, l_2, r_2, \dots, l_n, r_n}(t_2), & \begin{array}{l} \text{if for each } i: p_{i, q_i} = l_{i, r_i} \in L_i, \\ \text{and } t = t_2 \in T_2 - T_1, \\ \text{or if } t = t_2 \in T_1 \cup T_2, \text{ for almost all } i: \\ p_{i, q_i} = l_{i, r_i} \in L_i \\ \text{and there is } i \text{ such that } l_{i, r_i} \notin K_i; \end{array} \\ a_{k_1, s_1, k_2, s_2, \dots, k_n, s_n}(\#(t_2, T_1)) \circ b_{l_1, r_1, l_2, r_2, \dots, l_n, r_n}(\#(t_1, T_2)), & \begin{array}{l} \text{if for each } i: p_{i, q_i} = k_{i, s_i} = l_{i, r_i} \in \\ K_i \cap L_i \text{ and } t_1 \in T_1, t_2 \in T_2; \end{array} \\ \perp, & \text{otherwise.} \end{array} \right.$$

Operation “Termwise multiplication”

$$\begin{aligned} & A(T_1) \otimes_{(\circ)}^{[*]} B(T_2) \\ &= [K_1 \cap L_1, K_2 \cap L_2, \dots, K_n \cap L_n, T_1 \cap T_2, \{c_{p_1, q_1, p_2, q_2, \dots, p_n, q_n}(t)\}], \end{aligned}$$

where

$$\begin{aligned} & c_{p_1, q_1, p_2, q_2, \dots, p_n, q_n}(t) \\ &= a_{k_1, s_1, k_2, s_2, \dots, k_n, s_n}(\#(t_2, T_1)) \circ b_{l_1, r_1, l_2, r_2, \dots, l_n, r_n}(\#(t_1, T_2)) \end{aligned}$$

for each i , $p_{i, q_i} = k_{i, s_i} = l_{i, r_i} \in K_i \cap L_i$ and $t_1 \in T_1, t_2 \in T_2$.

Operation “Multiplication”

$$\begin{aligned} & A(T_1) \odot_{(\circ, *, \alpha, \beta)} B(T_2) \\ &= [K_1, \dots, K_\alpha, K_{n-\beta+1} \cup L_1, \dots, K_n \cup L_\beta, L_{n-\alpha+1}, \dots, L_n, T_1 \cup T_2, \\ & \quad \{c_{p_1, q_1, p_2, q_2, \dots, p_{2\alpha+\beta}, q_{2\alpha+\beta}}(t)\}], \end{aligned}$$

where

$$\begin{aligned}
& c_{p_1, q_1, p_2, q_2, \dots, p_{2\alpha+\beta}, q_{2\alpha+\beta}}(t) \\
= & \left\{ \begin{array}{ll} \circ & (a_{k_1, s_1, k_2, s_2, \dots, k_n, s_n}(\tau_{\max}(t_2, T_1))) \text{ if } p_{1, q_1} = k_{1, s_1} \in K_1, \\ & k_{\alpha+1, s_{\alpha+1}} = l_{\beta+1, t_{\beta+1}} \in K_{\alpha+1} \cap L_{\beta+1} \quad \dots \\ & k_{n-\beta, s_{n-\beta}} = l_{n-\alpha, t_{n-\alpha}} \in K_{n-\beta} \cap L_{n-\alpha} \quad p_{\alpha, q_\alpha} = k_{\alpha, s_\alpha} \in K_\alpha, \\ & * \quad b_{l_1, r_1, l_2, r_2, \dots, l_n, r_n}(\tau_{\max}(t_1, T_2)), \quad p_{\alpha+1, q_{\alpha+1}} = k_{n-\beta+1, s_{n-\beta+1}} \\ & \quad \quad \quad = l_{1, t_1} \in K_{n-\beta+1} \cap L_1, \dots \\ & \quad \quad \quad p_{\alpha+\beta, q_{\alpha+\beta}} = k_{n, s_n} \\ & \quad \quad \quad = l_{\beta, r_\beta} \in K_n \cap L_\beta, \\ & \quad \quad \quad p_{\alpha+\beta+1, q_{\alpha+\beta+1}} \\ & \quad \quad \quad = l_{n-\alpha+1, r_{n-\alpha+1}} \in \\ & \quad \quad \quad L_{n-\alpha+1}, \dots \\ & \quad \quad \quad p_{2\alpha+\beta, q_{2\alpha+\beta}} = \\ & \quad \quad \quad l_{n, t_n} \in L_n \\ & \quad \quad \quad \text{and } t_1 \in T_1, t_2 \in T_2; \\ \perp & \text{otherwise.} \end{array} \right.
\end{aligned}$$

Operation $\circ$ with a constant

$$\begin{aligned}
\alpha(T_1)(\circ, \#)A(T_2) = & [K_1, K_2, \dots, K_n, T_1 \cup T_2, \\ & \{\alpha(\#(t_2, T_1)) \circ a_{k_1, s_1, k_2, s_2, \dots, k_n, s_n}(\#(t_1, T_2))\}].
\end{aligned}$$

6.8 Relations over n -DTEIMs

Let the two n -DTEIMs $A(T_1)$ and $B(T_2)$ be given. We will introduce the following definitions where $\subset$ and $\subseteq$ denote the relations “*strong inclusion*” and “*weak inclusion*”.

The strict relation “Inclusion about dimension”

$$\begin{aligned}
A(T_1) \subset_d B(T_2) \text{ iff } & ((\forall i)(K_i \subset L_i) \vee (\forall i)(K_i \subseteq L_i \ \& \ (\exists j : 1 \leq j \leq n)(K_j \neq L_j)) \\ & \& (\forall i)(\forall j : 1 \leq s_j \leq m_i) \\ & (a_{k_1, s_1, k_2, s_2, \dots, k_n, s_n} = b_{k_1, s_1, k_2, s_2, \dots, k_n, s_n}) \ \& \ (T_1 = T_2)).
\end{aligned}$$

The non-strict relation “Inclusion about dimension”

$$\begin{aligned}
A(T_1) \subseteq_d B(T_2) \text{ iff } & ((\forall i)(K_i \subseteq L_i) \ \& \ (\forall i)(\forall j : 1 \leq s_j \leq m_i) \\ & (a_{k_1, s_1, k_2, s_2, \dots, k_n, s_n} = b_{k_1, s_1, k_2, s_2, \dots, k_n, s_n}) \ \& \ (T_1 = T_2)).
\end{aligned}$$

The strict relation “Inclusion about value”

$$\begin{aligned}
A(T_1) \subset_v B(T_2) \text{ iff } & (\forall i)(K_i = L_i) \ \& \ (\forall i)(\forall s_i : 1 \leq s_i \leq m_i) \\ & (a_{k_1, s_1, k_2, s_2, \dots, k_n, s_n} < b_{k_1, s_1, k_2, s_2, \dots, k_n, s_n}) \ \& \ (T_1 = T_2).
\end{aligned}$$

The non-strict relation “Inclusion about value”

$$A(T_1) \subset_v B(T_2) \text{ iff } (\forall i)(K_i = L_i) \& (\forall i)(\forall s_i : 1 \leq s_i \leq m_i) \\ (a_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}} \leq b_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}}) \& (T_1 = T_2).$$

The strict relation “Inclusion about temporal scale”

$$A(T_1) \subseteq_t B(T_2) \text{ iff } ((\forall i)(K_i = L_i) \& (\forall i)(\forall j : 1 \leq s_j \leq m_i) \\ (a_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}} = b_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}}) \& (T_1 \subset T_2)).$$

The non-strict relation “Inclusion about temporal scale”

$$A(T_1) \subseteq_t B(T_2) \text{ iff } ((\forall i)(K_i = L_i) \& (\forall i)(\forall j : 1 \leq s_j \leq m_i) \\ (a_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}} = b_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}}) \& (T_1 \subseteq T_2)).$$

The strict relation “Inclusion”

$$A(T_1) \subset B(T_2) \text{ iff } ((\forall i)(K_i \subset L_i) \vee (\forall i)(K_i \subseteq L_i \& (\exists j : 1 \leq s_j \leq n)(K_j \neq L_j)) \\ \& (\forall i)(\forall j : 1 \leq j \leq m_i) \\ (a_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}} < b_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}}) \& (T_1 \subset T_2)).$$

The non-strict relation “Inclusion”

$$A(T_1) \subseteq B(T_2) \text{ iff } (\forall i)(K_i \subseteq L_i) \& (\forall i)(\forall j : 1 \leq s_j \leq m_i) \\ (a_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}} \leq b_{k_{1,s_1}, k_{2,s_2}, \dots, k_{n,s_n}}) \& (T_1 \subseteq T_2).$$

Predicate index matrices

The present chapter contains some new results related to a new direction of development of IM theory. First, we will describe Predicate IMs (PIMs) with classical evaluations and after this we will discuss the PIMs with intuitionistic fuzzy evaluations.

7.1 Predicate index matrices with classical evaluations

In this section, extending [30], we will discuss three forms of PIMs.

First, let us have a set of indices $\mathcal{I}$ and a set of predicates $\mathcal{P}$ such that to each index to juxtapose a predicate. Let $V : \mathcal{P} \rightarrow \{false, true\}$ be an evaluation function for predicates. Then we can construct a PIM that has the form:

$$[K^{\textcircled{a}}, L^{\textcircled{a}}, \{a_{k_i, l_j}\}] \equiv \begin{array}{c|ccccc} & \langle l_1, R_1^L \rangle & \dots & \langle l_j, R_j^L \rangle & \dots & \langle l_n, R_n^L \rangle \\ \hline \langle k_1, R_1^K \rangle & a_{k_1, l_1} & \dots & a_{k_1, l_j} & \dots & a_{k_1, l_n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \langle k_i, R_i^K \rangle & a_{k_i, l_1} & \dots & a_{k_i, l_j} & \dots & a_{k_i, l_n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \langle k_m, R_m^K \rangle & a_{k_m, l_1} & \dots & a_{k_m, l_j} & \dots & a_{k_m, l_n} \end{array},$$

where for $1 \leq i \leq m$, and $1 \leq j \leq n : a_{k_i, l_j} \in \mathcal{X}$

$$K = \{k_1, k_2, \dots, k_m\} \subseteq \mathcal{I}, \quad L = \{l_1, l_2, \dots, l_n\} \subseteq \mathcal{I},$$

$$\{R_1^K, R_2^K, \dots, R_m^K, R_1^L, R_2^L, \dots, R_n^L\} \subseteq \mathcal{P},$$

$$K^{\textcircled{a}} = \{\langle k_i, R_i^K \rangle | 1 \leq i \leq m\}, \quad L^{\textcircled{a}} = \{\langle l_j, R_j^L \rangle | 1 \leq j \leq n\}.$$

Let $\#_1, \#_2, \circ, * : \mathcal{P} \times \mathcal{P} \rightarrow \mathcal{P}$ be operations over predicates, e.g., $\#_1, \#_2, \circ, * \in \{\wedge, \vee, \rightarrow, \equiv, \dots\}$ and let $e_0 \in \{false, true\}$.

Now, the operations over two PIMs are the following.

Strong addition

$$A \oplus_{(\#_1; \#_2; \circ)} B = [K^{\textcircled{a}} \cup^{\textcircled{a}} P^{\textcircled{a}}, L^{\textcircled{a}} \cup^{\textcircled{a}} Q^{\textcircled{a}}, \{c_{t_u, v_w}\}],$$

where operation $\cup^{\textcircled{a}}$ is defined by:

$$\begin{aligned} K^{\textcircled{a}} \cup^{\textcircled{a}} P^{\textcircled{a}} &= \{\langle c_{t_u}, R_{t_u}^K \#_1 R_{t_u}^P \rangle \mid t_u \in K \cup P\}, \\ L^{\textcircled{a}} \cup^{\textcircled{a}} Q^{\textcircled{a}} &= \{\langle d_{v_w}, R_{t_u}^L \#_2 R_{t_u}^Q \rangle \mid t_u \in K \cup P\}, \end{aligned}$$

and

$$c_{t_u, v_w} = \begin{cases} a_{k_i, l_j}, & \text{if } t_u = k_i \in K \text{ and } v_w = l_j \in L - Q \\ & \text{or } t_u = k_i \in K - P \text{ and } v_w = l_j \in L; \\ b_{p_r, q_s}, & \text{if } t_u = p_r \in P \text{ and } v_w = q_s \in Q - L \\ & \text{or } t_u = p_r \in P - K \text{ and } v_w = q_s \in Q; \\ a_{k_i, l_j} \circ b_{p_r, q_s}, & \text{if } t_u = k_i = p_r \in K \cap P \\ & \text{and } v_w = l_j = q_s \in L \cap Q \\ & \text{and } V(R_{t_u}^K) = V(R_{t_u}^P) = V(R_{t_u}^L) \\ & \quad = V(R_{t_u}^Q) = \text{true}; \\ e_0, & \text{otherwise.} \end{cases}$$

Weak addition

$$A \oplus_{(\#_1; \#_2; \circ)} B = [K^{\textcircled{a}} \cup^{\textcircled{a}} P^{\textcircled{a}}, L^{\textcircled{a}} \cup^{\textcircled{a}} Q^{\textcircled{a}}, \{c_{t_u, v_w}\}],$$

where $K^{\textcircled{a}} \cup^{\textcircled{a}} P^{\textcircled{a}}$ and $L^{\textcircled{a}} \cup^{\textcircled{a}} Q^{\textcircled{a}}$ have the above forms and

$$c_{t_u, v_w} = \begin{cases} a_{k_i, l_j}, & \text{if } t_u = k_i \in K \text{ and } v_w = l_j \in L - Q \\ & \text{or } t_u = k_i \in K - P \text{ and } v_w = l_j \in L; \\ b_{p_r, q_s}, & \text{if } t_u = p_r \in P \text{ and } v_w = q_s \in Q - L \\ & \text{or } t_u = p_r \in P - K \text{ and } v_w = q_s \in Q; \\ a_{k_i, l_j} \circ b_{p_r, q_s}, & \text{if } t_u = k_i = p_r \in K \cap P \\ & \text{and } v_w = l_j = q_s \in L \cap Q \\ & \text{and } V(R_{t_u}^K \vee R_{t_u}^P) = V(R_{t_u}^L \vee R_{t_u}^Q) = \text{true}; \\ e_0, & \text{otherwise.} \end{cases}$$

Strong termwise multiplication

$$A \otimes_{(\#_1; \#_2; \circ)} B = [K^{\textcircled{a}} \cap^{\textcircled{a}} P^{\textcircled{a}}, L^{\textcircled{a}} \cap^{\textcircled{a}} Q^{\textcircled{a}}, \{c_{t_u, v_w}\}],$$

where

$$\begin{aligned} K^{\textcircled{a}} \cap^{\textcircled{a}} P^{\textcircled{a}} &= \{\langle c_{t_u}, R_{t_u}^K \#_1 R_{t_u}^P \rangle \mid t_u \in K \cap P\}, \\ L^{\textcircled{a}} \cap^{\textcircled{a}} Q^{\textcircled{a}} &= \{\langle d_{v_w}, R_{t_u}^L \#_2 R_{t_u}^Q \rangle \mid t_u \in K \cap P\}, \end{aligned}$$

and

$$c_{t_u, v_w} = \begin{cases} a_{k_i, l_j} \circ b_{p_r, q_s}, & \text{if } V(R_{t_u}^K) = V(R_{t_u}^P) = V(R_{t_u}^L) = V(R_{t_u}^Q) = \text{true}; \\ e_0, & \text{otherwise} \end{cases}$$

for $t_u = k_i = p_r \in K \cap P$ and $v_w = l_j = q_s \in L \cap Q$.

Weak termwise multiplication

$$A \otimes_{(\#_1; \#_2; \circ)} B = [K^{\textcircled{a}} \cap^{\textcircled{a}} P^{\textcircled{a}}, L^{\textcircled{a}} \cap^{\textcircled{a}} Q^{\textcircled{a}}, \{c_{t_u, v_w}\}],$$

where $K^{\textcircled{a}} \cap^{\textcircled{a}} P^{\textcircled{a}}$ and $L^{\textcircled{a}} \cap^{\textcircled{a}} Q^{\textcircled{a}}$ have the above forms and

$$c_{t_u, v_w} = \begin{cases} a_{k_i, l_j} \circ b_{p_r, q_s}, & \text{if } V(R_{t_u}^K \vee R_{t_u}^P) = V(R_{t_u}^L \vee R_{t_u}^Q) = \text{true}; \\ e_0, & \text{otherwise} \end{cases}$$

for $t_u = k_i = p_r \in K \cap P$ and $v_w = l_j = q_s \in L \cap Q$.

Strong multiplication

$$A \odot_{(\#_1; \#_2; \circ, *)} B = [K^{\textcircled{a}} \cup^{\textcircled{a}} (P^{\textcircled{a}} - L^{\textcircled{a}}), Q^{\textcircled{a}} \cup^{\textcircled{a}} (L^{\textcircled{a}} - P^{\textcircled{a}}), \{c_{t_u, v_w}\}],$$

where $K^{\textcircled{a}} \cup^{\textcircled{a}} (P^{\textcircled{a}} - L^{\textcircled{a}})$ and $Q^{\textcircled{a}} \cup^{\textcircled{a}} (L^{\textcircled{a}} - P^{\textcircled{a}})$ have the above forms and

$$c_{t_u, v_w} = \begin{cases} a_{k_i, l_j}, & \text{if } t_u = k_i \in K \text{ and } v_w = l_j \in L - P - Q; \\ b_{p_r, q_s}, & \text{if } t_u = p_r \in P - L - K \text{ and } v_w = q_s \in Q; \\ \begin{aligned} & \circ_{l_j = p_r \in L \cap P} (a_{k_i, l_j} * b_{p_r, q_s}), \text{ if } t_u = k_i \in K \text{ and } v_w = q_s \in Q \\ & \text{and for each } l_j = p_r \in L \cup P : \\ & V(R_{l_j}^L) = V(R_{p_r}^P) = \text{true}; \end{aligned} \\ e_0, & \text{otherwise.} \end{cases}$$

Weak multiplication

$$A \odot_{(\#; \circ, *)} B = [K^{\textcircled{a}} \cup^{\textcircled{a}} (P^{\textcircled{a}} - L^{\textcircled{a}}), Q^{\textcircled{a}} \cup^{\textcircled{a}} (L^{\textcircled{a}} - P^{\textcircled{a}}), \{c_{t_u, v_w}\}],$$

where $K^{\textcircled{a}} \cup^{\textcircled{a}} (P^{\textcircled{a}} - L^{\textcircled{a}})$ and $Q^{\textcircled{a}} \cup^{\textcircled{a}} (L^{\textcircled{a}} - P^{\textcircled{a}})$ have the above forms and

$$c_{t_u, v_w} = \begin{cases} a_{k_i, l_j}, & \text{if } t_u = k_i \in K \text{ and } v_w = l_j \in L - P - Q; \\ b_{p_r, q_s}, & \text{if } t_u = p_r \in P - L - K \text{ and } v_w = q_s \in Q; \\ \begin{aligned} & \circ_{l_j = p_r \in L \cap P} (a_{k_i, l_j} * b_{p_r, q_s}), \text{ if } t_u = k_i \in K \text{ and } v_w = q_s \in Q \\ & \text{and for each } l_j = p_r \in L \cup P : \\ & V(R_{l_j}^L \vee R_{p_r}^P) = \text{true}; \end{aligned} \\ e_0, & \text{otherwise} \end{cases}$$

The remaining operations are as in Chapter 1.

Second, let $\{P_{k_i, l_j} | k_i \in K, l_j \in L\}$ be a set of predicates and $\{a_{k_i, l_j} | k_i \in K, l_j \in L\}$ be a set of objects, related to the respective predicates. Then we can construct the following $(0, 1)$ - or $(false, true)$ -IM (see Section 3.6 for the case $n = 2$):

$$\begin{aligned}
& [K, L, \{P_{k_i, l_j}(a_{k_i, l_j})\}] \\
&= \begin{array}{c|cccccc} & l_1 & \dots & l_j & \dots & l_n \\ \hline k_1 & P_{k_1, l_1}(a_{k_1, l_1}) & \dots & P_{k_1, l_j}(a_{k_1, l_j}) & \dots & P_{k_1, l_n}(a_{k_1, l_n}) \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_i & P_{k_i, l_1}(a_{k_i, l_1}) & \dots & P_{k_i, l_j}(a_{k_i, l_j}) & \dots & P_{k_i, l_n}(a_{k_i, l_n}) \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_m & P_{k_m, l_1}(a_{k_m, l_1}) & \dots & P_{k_m, l_j}(a_{k_m, l_j}) & \dots & P_{k_m, l_n}(a_{k_m, l_n}) \end{array} \\
&= \begin{array}{c|cccccc} & l_1 & \dots & l_j & \dots & l_n \\ \hline k_1 & P_{k_1, l_1} & \dots & P_{k_1, l_j} & \dots & P_{k_1, l_n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_i & P_{k_i, l_1} & \dots & P_{k_i, l_j} & \dots & P_{k_i, l_n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_m & P_{k_m, l_1} & \dots & P_{k_m, l_j} & \dots & P_{k_m, l_n} \end{array} \oplus \begin{array}{c|cccccc} & l_1 & \dots & l_j & \dots & l_n \\ \hline k_1 & a_{k_1, l_1} & \dots & a_{k_1, l_j} & \dots & a_{k_1, l_n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_i & a_{k_i, l_1} & \dots & a_{k_i, l_j} & \dots & a_{k_i, l_n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_m & a_{k_m, l_1} & \dots & a_{k_m, l_j} & \dots & a_{k_m, l_n} \end{array}
\end{aligned}$$

where operation $\oplus$ is defined in Section 3.6.

Finally, we can construct a combination of the two types PIMs, discussed above in the form:

$$\begin{aligned}
& [K^{\otimes}, L^{\otimes}, \{P_{k_i, l_j}(a_{k_i, l_j})\}] \\
&= \begin{array}{c|cccccc} & \langle l_1, R_1^L \rangle & \dots & \langle l_j, R_j^L \rangle & \dots & \langle l_n, R_n^L \rangle \\ \hline \langle k_1, R_1^K \rangle & P_{k_1, l_1}(a_{k_1, l_1}) & \dots & P_{k_1, l_j}(a_{k_1, l_j}) & \dots & P_{k_1, l_n}(a_{k_1, l_n}) \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \langle k_i, R_i^K \rangle & P_{k_i, l_1}(a_{k_i, l_1}) & \dots & P_{k_i, l_j}(a_{k_i, l_j}) & \dots & P_{k_i, l_n}(a_{k_i, l_n}) \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \langle k_m, R_m^K \rangle & P_{k_m, l_1}(a_{k_m, l_1}) & \dots & P_{k_m, l_j}(a_{k_m, l_j}) & \dots & P_{k_m, l_n}(a_{k_m, l_n}) \end{array}
\end{aligned}$$

For the so constructed PIMs, the operations, relations and operators discussed in previous chapters will have similar forms.

7.2 Predicate IMs with intuitionistic fuzzy evaluations

Let $\mathcal{L} = \{\langle a, b \rangle | a, b, a + b \in [0, 1]\}$. Similarly to Section 5.4, let us have a set of indices $\mathcal{I}$ and a set of predicates $\mathcal{P}$ but now let $V : \mathcal{P} \rightarrow \mathcal{L}$ be an evaluation function for predicates (for $\mathcal{L}$ see Section 4.1). The new type of

PIM has the same form as in the previous section, but now there are changes in the definitions of the operations over two PIMs.

Let the IFP $\langle a, b \rangle$ be called

- a tautological IFP, when $a = 1, b = 0$,
- a sure IFP, when $1 \geq a \geq 0.5 \geq b \geq 0$,
- an Intuitionistic Fuzzy Tautological (IFT) IFP, when $1 \geq a \geq b \geq 0$.

Now, the definitions of the operations from Section 7.1 of Z-type, where Z is term “tautological”, “sure” or “intuitionistic fuzzy tautological”, are the following

Addition of Z-type

$$A \oplus_{(\#_1; \#_2; \circ)} B = [K^{\textcircled{a}} \cup^{\textcircled{a}} P^{\textcircled{a}}, L^{\textcircled{a}} \cup^{\textcircled{a}} Q^{\textcircled{a}}, \{c_{t_u, v_w}\}],$$

where $K^{\textcircled{a}} \cup^{\textcircled{a}} P^{\textcircled{a}}$ and $L^{\textcircled{a}} \cup^{\textcircled{a}} Q^{\textcircled{a}}$ are as above and

$$c_{t_u, v_w} = \begin{cases} a_{k_i, l_j}, & \text{if } t_u = k_i \in K \text{ and } v_w = l_j \in L - Q \\ & \text{or } t_u = k_i \in K - P \text{ and } v_w = l_j \in L; \\ b_{p_r, q_s}, & \text{if } t_u = p_r \in P \text{ and } v_w = q_s \in Q - L \\ & \text{or } t_u = p_r \in P - K \text{ and } v_w = q_s \in Q; \\ a_{k_i, l_j} \circ b_{p_r, q_s}, & \text{if } t_u = k_i = p_r \in K \cap P \text{ and } v_w = l_j = q_s \in L \cap Q \\ & \text{and } V(R_{t_u}^K), V(R_{t_u}^P), V(R_{t_u}^L), V(R_{t_u}^Q) \\ & \text{are of Z-type;} \\ e_0, & \text{otherwise.} \end{cases}$$

Termwise multiplication of Z-type

$$A \otimes_{(\#_1; \#_2; \circ)} B = [K^{\textcircled{a}} \cap^{\textcircled{a}} P^{\textcircled{a}}, L^{\textcircled{a}} \cap^{\textcircled{a}} Q^{\textcircled{a}}, \{c_{t_u, v_w}\}],$$

where $K^{\textcircled{a}} \cap^{\textcircled{a}} P^{\textcircled{a}}$ and $L^{\textcircled{a}} \cap^{\textcircled{a}} Q^{\textcircled{a}}$ have the above forms and

$$c_{t_u, v_w} = \begin{cases} a_{k_i, l_j} \circ b_{p_r, q_s}, & \text{if } V(R_{t_u}^K), V(R_{t_u}^P), V(R_{t_u}^L), V(R_{t_u}^Q) \\ & \text{are of Z-type;} \\ e_0, & \text{otherwise} \end{cases}$$

for $t_u = k_i = p_r \in K \cap P$ and $v_w = l_j = q_s \in L \cap Q$.

Multiplication of Z-type

$$A \odot_{(\#_1; \#_2; \circ, *)} B = [K^{\textcircled{a}} \cup^{\textcircled{a}} (P^{\textcircled{a}} - L^{\textcircled{a}}), Q^{\textcircled{a}} \cup^{\textcircled{a}} (L^{\textcircled{a}} - P^{\textcircled{a}}), \{c_{t_u, v_w}\}],$$

where $K^{\textcircled{a}} \cup^{\textcircled{a}} (P^{\textcircled{a}} - L^{\textcircled{a}})$ and $Q^{\textcircled{a}} \cup^{\textcircled{a}} (L^{\textcircled{a}} - P^{\textcircled{a}})$ have the above forms and

$$c_{t_u, v_w} = \begin{cases} a_{k_i, l_j}, & \text{if } t_u = k_i \in K \text{ and } v_w = l_j \in L - P - Q; \\ b_{p_r, q_s}, & \text{if } t_u = p_r \in P - L - K \text{ and } v_w = q_s \in Q; \\ l_j = p_r \overset{\circ}{\in} L \cap P (a_{k_i, l_j} * b_{p_r, q_s}), & \text{if } t_u = k_i \in K \text{ and } v_w = q_s \in Q \\ & \text{and for each } l_j = p_r \in L \cup P : \\ & V(R_{l_j}^L) \text{ and } V(R_{p_r}^P) \text{ are of } Z\text{-type}; \\ 0, & \text{otherwise} \end{cases}$$

The remaining operations are as in Chapter 1.

Conclusion

As it was mentioned in the Foreword, this book contains my main results on index matrices, obtained mainly after the publication of [20]. The book does not include the results of the remaining colleagues who worked on the theory of IMs like Velichka Traneva, my former PhD student with a thesis on IMs, as well as on IM applications in different areas of sciences—here the list is essentially longer, but the most active have been (in alphabetical order): Deyan Mavrov, Evdokia Sotirova, Lyudmila Todorova, Nora Angelova, Olympia Roeva, Pencho Marinov, Peter Vassilev, Simeon Ribagin, Sotir Sotirov, Stela Todorova, Tania Pencheva, Vassia Atanassova, Velin Andonov, Veselina Bureva, and others. The reason is that I hope at least some of them will follow my example and publish their results in subsequent books in near future.

Over the past years, IMs have been used as tools for describing of different processes and objects. For example, they are used in the areas of:

- generalized nets – [1, 2, 4, 9, 83],
- intuitionistic fuzzy sets, relations, operations, operators, graphs – [5–8, 13, 52, 56],
- neural networks – [46, 47, 70],
- image and pattern recognition – [66]
- game method for modelling – [12],
- databases, big data, OLAP [39, 40, 51, 54, 55, 57–59, 74, 75],
- decision making procedures – [42, 43, 61, 79–82],
- transportation problems – [76, 77],
- number theory – [23],

and others.

I would be very happy if the book succeeds in provoking scientific interest and stimulates other fellow researchers to start working in this area.

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Krassimir T. Atanassov

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Index matrices, developed over more than four decades, provide a novel framework for extending classical matrix calculus to handle otherwise ill-posed operations between matrices of differing dimensions. Originating from early theoretical insights and refined through years of research, they have found applications in generalized nets, graph theory, fuzzy systems, decision-making, and beyond. The book presents the core concepts, key results, and evolving extensions of index matrices, including multidimensional formulations and their associated operations. Through intuitive examples, it demonstrates how seemingly undefined operations can be transformed into mathematically consistent and meaningful ones. The author also highlights some open problems, inviting further exploration in this rich and growing area of research.

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